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PYTHAGOREAN PICTURE FUZZY SETS: PART 2-SOME MAIN PICTURE LOGIC OPERATORS AND PICTURE INFERENCE PROCESSES

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Abstract. Pythagorean picture fuzzy set (PPFS) is a combination of Picture Fuzzy Set and the Yager's Pythagorean Fuzzy Set [12–14]. In the first part of the paper [17] we introduced some basic notions namely the set operators on PPFS. The missing part in our previous work is defining the extension of such the operators on the Spherical Fuzzy Sets toward applications of multiattribute group decision making problems. In this second part, we will tackle this issue and present some main operators on PPFS such as the picture negation operator, picture t-norm, picture t-conorm, picture implication operators on PPFS. Lastly, the compositional rule of inference in PPFS is presented accompanied with a numerical example.

Keywords. Picture fuzzy set, Pythagorean picture fuzzy set, picture logic operators, decision-making problems.

1. INTRODUCTION

Bui Cong Cuong and Vladik Kreinovich firstly defined the concept of the picture fuzzy sets (PFS) in 2013 [8], which is the generalization of the Zadeh' fuzzy sets [1] and the Antanassov's intuitionistic fuzzy sets [3]. This concept is particularly effective in approaching the practical problems in relation to the synthesis of ideas, such as decisions making problems, voting analysis, fuzzy clustering, financial forecasting. The basic notions of the picture fuzzy sets theory were given in [9–11, 34, 35]. The connectives in picture fuzzy logic were also presented in [8] that supported to new computing procedures in computational intelligence and in other applications (see [17-33]).

In 2013 Yager introduced a new concept – Pythagorean fuzzy set (PFS, see [12–14]) with some applications in decision making. This paper is devoted to Pythagorean Picture Fuzzy set (PPFS) - a combination of Picture fuzzy set and the Pythagorean fuzzy set. Firstly, we present basic notions on PPFS such as set operators and Cartesian product of PPFS, Pythagorean picture relation, Pythagorean picture fuzzy soft set. Next, we will study some basic operators of the Picture Fuzzy Logic on PPFS.

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2. PRELIMINARY

2.1. Pythagorean picture fuzzy sets

We first recall the basic notions of the picture fuzzy sets.

Definition 2.1. [9] A picture fuzzy set A on a universe U is an object of the form

$$A = \{(u, x_{1A}(u),\ x_{2A}(u),\ x_{3A}(u))\, | u \in U\},$$

where $x_{1A}(u),\ x_{2A}(u),\ x_{3A}(u)$ are respectively called the degree of positive membership, the degree of neutral membership, the degree of negative membership of u in A, and the following conditions are satisfied

$$0 \leq x_{1A}(u), x_{2A}(u), x_{3A}(u) \leq 1 \quad \text{and} \quad 0 \leq x_{1A}(u) + x_{2A}(u) + x_{3A}(u) \leq 1,\ \forall u \in U.$$

Then, $\forall u \in U : x_{4A}(u) = 1 - (x_{1A}(u) + x_{2A}(u) + x_{3A}(u))$ is called the degree of refusal membership of u in A.

Definition 2.2. [17, 19] A Pythagorean picture fuzzy set (PPFS) A on a universe U is an object of the form $A = \{(u,\ x_{1A}(u),\ x_{2A}(u),\ x_{3A}(u))\, | u \in U\}$, where $x_{1A}(u),\ x_{2A}(u),\ x_{3A}(u)$ are respectively called the degree of positive membership, the degree of neutral membership, the degree of negative membership of u in A, and the following conditions are satisfied $0 \leq x_{1A}(u),\ x_{2A}(u),\ x_{3A}(u) \leq 1$ and $0 \leq x_{1A}^2(u) + x_{2A}^2(u) + x_{3A}^2(u) \leq 1,\ \forall u \in U$.

Consider the set $D^* = \left\{x = (x_1, x_2, x_3) | x \in [0{,}1]^3,\ x_1 + x_2 + x_3 \leq 1\right\}$, and

$$P^* = \{x = (x_1, x_2, x_3) | x \in [0{,}1]^3,\ x_1^2 + x_2^2 + x_3^2 \leq 1\}.$$

Denote $0_{D^*} = 0_{P^*} = (0, 0, 1) \in P^*,\ 1_{D^*} = 1_{P^*} = (1, 0, 0) \in P^*,\ D^* \subseteq P^*$.

From now on, we will assume that if $x \in P^*$, then $x_1,\ x_2$ and x_3 denote, respectively, the first, the second and the third component of x, i.e., $x = (x_1, x_2, x_3)$. Let $x, y \in P^*, y = (y_1, y_2, y_3)$.

We have a lattice $(P^*, \leq_1)$, where $\leq_1$ is defined by $\forall x, y \in P^*$

$$(x \leq_1 y) \Leftrightarrow (x_1 < y_1, x_3 \geq y_3) \vee (x_1 = y_1, x_3 > y_3) \vee (\{x_1 = y_1, x_3 = y_3, x_2 \leq y_2\}),$$

$$(x = y) \Leftrightarrow (x_1 = y_1,\ x_2 = y_2,\ x_3 = y_3),\ \forall x, y \in P^*.$$

We define the first, second and third projection mappings pr_1, pr_2 and pr_3 on P^*, as $pr_1(x) = x_1$ and $pr_2(x) = x_2$ and $pr_3(x) = x_3$, for all $x \in P^*$.

Note that, if for $x, y \in P^*$ neither $x \leq_1 y$ nor $y \leq_1 x$, then x and y are incomparable w.r.t $\leq_1$, denoted as $x \|_{\leq_1} y$.

For each $x, y \in P^*$, we define

$$\inf(x, y) = \begin{cases} \min(x, y), \text{ if } x \leq_1 y \text{ or } y \leq_1 x, \\ (x_1 \wedge y_1,\ 1 - x_1 \wedge y_1 - x_3 \vee y_3,\ x_3 \vee y_3), \text{ otherwise}, \end{cases}$$

$$\sup(x,y)=\begin{cases}\max(x,y), & \text{if } x\leq_1 y \text{ or } y\leq_1 x,\\ (x_1\vee y_1,0,x_3\wedge y_3), & \text{otherwise.}\end{cases}$$

Proposition 2.1. *With these operators $(P^{*},\leq_1)$ is a complete lattice.*

Proof. See [11].

Using this lattice, we easily see that every picture fuzzy set

$$A=\{(u,x_{1A}(u),x_{2A}(u),x_{3A}(u))|\,|u\in U\}\ ,$$

corresponds an $P^{*}-$ fuzzy set mapping [11], i.e., we have a mapping

$$A:U\to P^{*}:u\to\{(u,x_{1A}(u),x_{2A}(u),x_{3A}(u))\,|u\in U\}.$$

Interpreting picture fuzzy sets as $P^{*}-$fuzzy sets gives a way to increase flexibility in calculating with membership degrees, since the triple of numbers formed by the three degrees is an element of P^{*}, and often allows to have more compact formulas.

Let $PFS(U)$ denote the set of all the picture fuzzy set PFSs on a universe $U$and $PPFS(U)$ denote the set of all the Pythagorean picture fuzzy set PPFSs on a universe U. ■

Definition 2.3. For every two PPFSs A and B, the inclusion, union, intersection and complement are defined as follows:

- $A\subseteq B$ iff $(\forall u\in U,\ x_{1A}(u)\leq x_{1B}(u),\ x_{2A}(u)\leq x_{2B}(u),\ x_{3A}(u)\geq x_{3B}(u))$,
- $A=B$ iff $(A\subseteq B$ and $B\subseteq A)$,
- $A\cup B=\{(u,\ x_{1A}(u)\vee x_{1B}(u),\ x_{2A}(u)\wedge x_{2B}(u),\ x_{3A}(u)\wedge x_{3B}(u))\,|u\in U\}$,
- $A\cap B=\{(u,\ x_{1A}(u)\wedge x_{1B}(u),\ x_{2A}(u)\wedge x_{2B}(u),\ x_{3A}(u)\vee x_{3B}(u))\,|u\in U\}$,
- $A=A^{c}=\left\{\left(u,\ x_{3A}(u),\ \sqrt{1-\left(x_{1A}^{2}(u)+x_{2A}^{2}(u)+x_{3A}^{2}(u)\right)},\ x_{1A}(u)\right)|u\in U\right\}$.

Definition 2.4. Let two sets $U_1$and U_2 be two universes and let

$$A=\{(u,\ x_{1A}(u),\ x_{2A}(u),\ x_{3A}(u))\,|u\in U_1\},$$

and $B=\{(v,\ x_{1A}(v),\ x_{2A}(v),\ x_{3A}(v))\,|v\in U_2\}$, be two PPFS sets. We define the Cartesian product of these PPFS's

$$A\times B=\{((u,v),\ x_{1A}(u)\wedge x_{1B}(v),\ x_{2A}(u)\wedge x_{2B}(v),\ x_{3A}(u)\vee x_{3B}(v))\,|(u,v)\in U_1\times U_2\}.$$

We denote the set of all picture fuzzy sets over $X_1\times X_2$ by $PPFS(X_1\times X_2)$.

Theorem 2.1. *For every three universes $U_1,\ U_2,\ U_3$ and four PPFSs $O_1,O_2\in PPFS(U_1)$, $O_3\in PPFS(U_2)O_4\in PPFS(U_3)$* there hold the relations

$$O_1\times O_3=O_3\times O_1,$$

$$(O_1\times O_3)\times O_4=O_1\times(O_3\times O_4),$$

$$(O_1\cup O_2)\times O_3=(O_1\times O_3)\cup(O_2\times O_3),$$

$$(O_1\cap O_2)\times O_3=(O_1\times O_3)\cap(O_2\times O_3).$$

2.2. Pythagorean picture fuzzy relation

The Zadeh' composition rule of inference (see [6, 11, 15]) is a well-known method in approximation theory and in inference methods in fuzzy control theory. Intuitionistic fuzzy relations were received many results by researches (see [4,5]). In this section we shall present some preliminary results in Pythagorean picture fuzzy relations.

Let X, Y and Z be ordinary non-empty sets.

An extension of the results given in [6, 7] for PPFS is the following.

Definition 2.5. A Pythagorean picture fuzzy relation R is a Pythagorean picture fuzzy subset of $X \times Y$, given by

$$R = \{((x,y),\ z_{1R}(x,y),\ z_{2R}(x,y),\ z_{3R}(x,y))\,|x \in X, y \in Y)\}\,,$$

where $z_{1R} : X \times Y \to [0,1],\ z_{2R} : X \times Y \to [0,1],\ z_{3R} : X \times Y \to [0,1]$ satisfy the condition $0 \le z_{1R}^2(x,y) + z_{2R}^2(x,y) + z_{3R}^2(x,y) \le 1$ for every $(x,y) \in (X \times Y)$.

3. SOME MAIN PICTURE LOGIC OPERATORS ON PPFS

Consider the set $P^* = \left\{x = (x_1, x_2, x_3)|x \in [0{,}1]^3,\ x_1^2 + x_2^2 + x_3^2 \le 1\right\}$.

From now on, we will assume that if $x \in P^*$, then x_1, x_2 and x_3 denote, respectively, the first, the second and the third component of x, and $x = (x_1, x_2, x_3)$ is called a Pythagorean picture fuzzy number.

Now we consider some basic connectives of the Picture Fuzzy Logic on PPFS.

3.1. Picture negation on PPFS

Picture negations on PPFS form an extension of the fuzzy negations [2, 5] and the intuitionistic fuzzy negations [4]. They are defined as follows.

Definition 3.1. A mapping $N : P^* \to P^*$ satisfying conditions $N(0_{P^*}) = 1_{P^*}$ and $N(1_{P^*}) = 0_{P^*}$ and N is nonincreasing is called a picture negation operator on PPFSs.

If $N(N(x)) = x$ for all $x \in P^*$, then N is called an *involutive* negation operator.

Definition 3.2. Let $f_1, f_2 : P^* \to D^*$ be mappings on D^*. We say that the mapping f_2 is greater than f_1 if $f_1(x) \le_1 f_2(x),\ \forall x \in P^*$, then we denote $f_1 \le f_2$. We write $f_1 < f_2$, if $f_1 \le f_2$ and $f_1 \ne f_2$.

Let $x = (x_1, x_2, x_3) \in P^*$. We first give 2 drastic picture negation operators on PPFS

$$n_d\,(x) = \begin{cases} 0_{P^*} & \text{if } \ x \ne 0_{P^*} \\ 1_{P^*} & \text{if } \ x = 0_{P^*}, \end{cases} \qquad n_{d2}\,(x) = \begin{cases} 1_{P^*} & \text{if } \ x \ne 1_{P^*} \\ 0_{P^*} & \text{if } \ x = 1_{P^*}. \end{cases}$$

Proposition 3.1. *The operators n_d and n_{d2} are picture negation operators on PPFS and for each picture negation operator $n(x)$ on PPFS $n_d(x) \le_1 n(x) \le_1 n_{d2}(x),\ \forall x \in P^*$.*

Definition 3.3. The mapping $n_0 : P^* \to P^*$ is defined by $n_0(x) = (x_3, 0, x_1)$, for each $x \in P^*$.

Proposition 3.2. *n_0 is a picture negation operator on PPFS. It is called the simple picture negation.*

Proof. See the proof in page 145 [11].

Definition 3.4. Let $x = (x_1, x_2, x_3) \in P^*$. Denote $z_4 = \sqrt{1 - (x_1^2 + x_2^2 + x_3^2)}$. The mapping N_S is defined by $N_S(x) = (x_3, z_4, x_1)$, for each $x \in P^*$.

Proposition 3.3. *N_S is an involutive picture negation operator on PPFS.*

Proof.

Indeed, $N_S(x) = (x_3, z_4, x_1) = (x_3, \sqrt{1 - (x_1^2 + x_2^2 + x_3^2)}, x_1) \Rightarrow$

$$\begin{aligned}
N_S(N_S(x)) &= N_S(x_3, \sqrt{1 - (x_1^2 + x_2^2 + x_3^2)},\ x_1) \\
&= (x_1, \sqrt{1 - (x_1^2 + (\sqrt{1 - (x_1^2 + x_2^2 + x_3^2)}))^2 + x_3^2},\ x_3) \\
&= (x_1, \sqrt{1 - (x_1^2 + (\sqrt{1 - x_1^2 - x_2^2 - x_3^2})^2 + x_3^2},\ x_3) \\
&= (x_1, \sqrt{x_2^2},\ x_3) \\
&= x.
\end{aligned}$$

■

3.2. Picture t-norms and t-conorms on PPFS

Fuzzy t-norms on [0,1] and fuzzy t-conorms on [0,1] which are important connectives in fuzzy logic were defined and considered in [2] (see also in [5,6]).

Now we define picture t-norms and picture t-conorms on PPFSs, which are classes of conjunction operators and classes of disjunction operators - main basic operators on PPFS of the picture fuzzy logics. Picture fuzzy t-norms are direct extensions of the fuzzy t-norms in [2,5,6] and of the intuitionistic fuzzy t-norms in [4]. Some simple t-norms and t-conorms were firstly given in [17].

Let $x = (x_1, x_2, x_3) \in P^*$. Denote $I(x) = \left\{ y \in D^* : y = (x_1^2,\ y_2,\ x_3^2),\ 0 \leq_1 y_2 \leq_1 x_2^2 \right\}$.

Definition 3.5. A mapping $T : P^* \times P^* \to D^*$ is a *picture t-norm* on PPFSs if the mapping T satisfies the following conditions:

1) $T(x, y) = T(y, x),\ \forall x, y \in \mathrm{P}^*$ (commutative);

2) $T(x, T(y, z)) = T(T(x, y), z),\ \forall x, y, z \in P^*$ (associativity);

3) $T(x, y) \leq_1 T(x, z),\ \forall x, y, z \in P^*,\ y \leq_1 z$ (monotonicity);

4) $T(1_{P^*}, x) \in I(x),\ \forall x \in P^*$ (boundary condition).

First we present some picture t-norms on Pythagorean picture fuzzy sets.

For all $x, y \in P^*$:

$$T_{\min}(x,y) = \left(x_1^2 \wedge y_1^2, x_2^2 \wedge y_2^2, x_3^2 \vee y_3^2\right).$$
$$T_2(x,y) = \left(x_1^2 \wedge y_1^2, x_2^2 y_2^2, \max\left(x_3^2, y_3^2\right)\right).$$
$$T_3(x,y) = \left(x_1^2 y_1^2, x_2^2 y_2^2, \max\left(x_3^2, y_3^2\right)\right).$$
$$T_4(x,y) = \left(x_1^2 y_1^2, x_2^2 y_2^2, x_3^2 + y_3^2 - x_3^2 y_3^2\right).$$

$$T_5(x,y) = \left(\begin{cases} x_1^2 \wedge y_1^2 & \text{if } \ x_1 \vee y_1 \ = 1 \\ 0 & \text{if } \ x_1 \vee y_1 \ < 1 \end{cases}, \begin{cases} l x_2^2 \wedge y_2^2 & \text{if } \ x_2 \vee y_2 = 1 \\ 0 & \text{if } \ x_2 \vee y_2 < 1 \end{cases}, \begin{cases} x_3^2 \vee y_3^2 & \text{if } \ x_3 \wedge y_3 = 0 \\ 1 & \text{if } \ x_3 \wedge y_3 \neq 0 \end{cases} \right).$$

Theorem 3.1. *The mappings $T_{\min}(x,y),\ T_2(X,y),\ T_3(x,y),\ T_4(x,y),\ T_5(x,y)$ are picture t - norms on PPFS.*

Proof. Now we give some detail proofs of picture t - norms on PPFS.

Let $x, y \in P^*,\ x = (x_1, x_2, x_3),\ y = (y_1, y_2, y_3)$.

- The mapping $T_{\min}$ is a picture fuzzy t-norm on PPFS.

Since $x, y \in P^*,$ then $x_1^2 + x_2^2 \leq 1 - x_3^2,$ and $y_1^2 + y_2^2 \leq 1 - y_3^2,$

$$(x_1^2 \wedge y_1^2) + (x_2^2 \wedge y_2^2) \leq \min(1 - x_3^2, 1 - y_3^2) = 1 - \max(x_3^2, y_3^2),$$

$$(x_1^2 \wedge y_1^2) + (x_2^2 \wedge y_2^2) + \max(x_3^2, y_3^2) \leq 1,$$

$$nT_{\min}(x,y) = ((x_1^2 \wedge y_1^2), (x_2^2 \wedge y_2^2), \max(x_3^2, y_3^2)) \in D^*.$$

- The mapping $T_2(x,y),\ T_3(x,y)$ are picture fuzzy t-norms on PPFS. Indeed, we remark that $x_2^2.y_2^2 \leq x_2^2 \wedge y_2^2 \Rightarrow \left((x_1^2 \wedge y_1^2) + x_2^2.y_2^2 + \max(x_3^2, y_3^3)\right) \leq T_{\min}(x,y) \in D^*$.

- For $T_3(x,y)$ we use the similar argument.

- The mapping $T_4(x,y) = \left(x_1^2 y_1^2,\ x_2^2 y_2^2,\ x_3^2 + y_3^2 - x_3^2 y_3^2\right)$ is a picture fuzzy t-norm.

We have

$$\begin{aligned} l x_1^2 y_1^2 + x_2^2 y_2^2 &\leq \left(1 - x_2^2 - x_3^2\right)\left(1 - y_2^2 - y_3^2\right) + x_2^2 y_2^2 \\ &= \left(1 - y_2^2 - y_3^2 - x_2^2 + x_2^2 y_3^2 + x_2^2 y_3^2 - x_3^2 + x_3^2 y_2^2 + x_3^2 y_3^2\right) + x_2^2 y_2^2 \\ &= \left(1 - x_3^2 - y_3^2 + x_3^2 y_3^2\right) + \left(x_2^2 y_2^2 + x_2^2 y_3^2 + x_2^2 y_2^2 + x_3^2 y_2^2 - x_2^2 - y_2^2\right) \\ &= \left(1 - x_3^2 - y_3^2 + x_3^2 y_3^2\right) + \left(x_2^2\left(y_2^2 + y_3^2 - 1\right) + y_2^2\left(x_2^2 + x_3^2 - 1\right)\right) \\ &\leq 1 - x_3^2 - y_3^2 + x_3^2 y_3^2. \end{aligned}$$

$\Rightarrow x_1^2 y_1^2 + x_2^2 y_2^2 + x_3^2 + y_3^2 - x_3^2 y_3^2 \leq 1.$

To prove that $T_5(x,y)$ is a is a picture t-norm, we remark that

$$g(x,y) = \begin{pmatrix} x_1^2 \wedge y_1^2 & \text{if} & x_1 \vee y_1 = 1 \\ 0 & \text{if} & x_1 \vee y_1 < 1 \end{pmatrix} \leq x_1^2 \wedge y_1^2,$$

$$g(x,y) = \begin{pmatrix} x_2^2 \wedge y_2^2 & \text{if} & x_2 \vee y_2 = 1 \\ 0 & \text{if} & x_2 \vee y_2 < 1 \end{pmatrix} \leq x_2^2 \wedge y_2^2.$$

∎

Proposition 3.4. *Let mapping t_2 is a fuzzy t-norm on [0,1]. Then the mapping*

$$T_{t_2}(x,y) = \left(\min\left(x_1^2, y_1^2\right),\ t_2(x_2^2, y_2^2),\ \max\left(x_3^2, y_3^2\right)\right)$$

is a picture fuzzy t - norm on PPFS.

Proof. See the proof of the given argument.

Definition 3.6. A mapping $S : P^* \times P^* \to D^*$ is a *picture t-conorm* on PPFS if Ssatisfies all following conditions

1) $S(x,y) = S(y,x)\,,\ \forall x,y \in P^*$.
2) $S(x, S(y,z)) = S(S(x,y), z)\,,\ \forall x,y,z \in P^*$.
3) $S(x,y) \leq_1 S(x,z)\,,\ \forall x,y,z \in P^*, y \leq_1 z$.
4) $S(x, 0_{P^*}) \in I(x)\,,\ \forall x \in P^*$.

Below are some examples of picture t-conorms on PPFS:

1) $S_{\max}(x,y) = \left(\max\left(x_1^2, y_1^2\right), \min\left(x_2^2, y_2^2\right), \min\left(x_3^2, y_3^2\right)\right)$.
2) $S_2(x,y) = \left(\max\left(x_1^2, y_1^2\right),\ x_2^2 y_2^2,\ \min\left(x_3^2, y_3^2\right)\right)$.
3) $S_3(x,y) = \left(\max\left(x_1^2, y_1^2\right),\ x_2^2 y_2^2,\ x_3^2 y_3^2\right)$.
4) $S_4(x,y) = \left(x_1^2 + y_1^2 - x_1^2 y_1^2,\ x_2^2 y_2^2,\ x_3^2 y_3^2\right)$.
5) $S_5(x,y) = \left(x_1^2 \vee y_1^2,\ \begin{cases} x_2^2 \wedge y_2^2 & \text{if} \quad x_2 \vee y_2 = 1 \\ 0 & \text{if} \quad x_2 \vee y_2 < 1 \end{cases},\ x_3^2 \wedge y_3^2\right)$.

3.3. Some classes of picture implications for PPFS

In this section we present some classes of picture implications for Pythagorean picture fuzzy sets, which are the direct generalizations of the classical implication operators and some classes of the fuzzy implication operators (see, for example [2, 4–7].

First important class of picture implication operators on PPFS is the followings.

Let $a, b \in P^*,\ a = (a_1, a_2, a_3),\ b = (b_1, b_2, b_3)$.

Definition 3.7. A mapping $I : P^* \times P^* \to D^*$ is a picture implication operator of the class 1 if it satisfies the following boundary conditions:

$I(0_{P^*},\ 1_{P^*}) = 1_{D^*}$, where $0_{P^*} = 0_{D^*} = (0,0,1),\ 1_{P^*} = 1_{D^*} = (1,0,0)$. (2.1)

$I(0_{P^*},\ 0_{P^*}) = 1_{D^*},\ I(1_{P^*}, 1_{P^*}) = 1_{D^*}$. (2.2) –(2.3)

$I(1_{P^*},\ 0_{P^*}) = 0_{P^*}$. (2.4)

Clearly, this definition of picture implication is a direct generalization of the classical implication and the definition of fuzzy implication operators given in [5].

Another class of the picture implications is defined in the following.

Definition 3.8. A mapping $I : P^* \times P^* \to D^*$ is a picture implication of the class 2 if it satisfies the following boundary conditions (2.1) - (2.4) and

$I(a_1,\ b) \geq_1 I(a_2, b),\ \forall\ a_1 \leq_1 a_2,\ b \in P^*$. (2.5)

$I(a,\ b_1) \leq_1 I(a, b_2),\ \forall\ b_1 \leq_1 b_2,\ a \in P^*$. (2.6)

This definition is a direct generalization of the definition 1.15 of the fuzzy implication operators given in [7, p. 22].

Proposition 3.5. *A picture implication operator of the class 2 is a picture implication operator of the class 1.*

Now we give some direct generalizations of the fuzzy implication operators.

Definition 3.9. Let $n(x)$ be a picture negation operator and let $S(x, y)$ be a picture fuzzy t-conorm operator.

A mapping $I : P^* \times P^* \to D*$ is given by

$$I(a, b) = S(n(a), b), \ \forall a, b \in P^*. \tag{2.7}$$

It is a new direct generalization of the fuzzy implications given in the Definition 6.1.3 [5, p.146].

Proposition 3.6. *The picture implication operators defined in the Definition 3.9 are picture implication operators of the class 2.*

Proof. Indeed, if

$$a = 0_{D^*} \Rightarrow I(0_{D^*}, b) = S(1_{D^*}, b), \text{ then } I(0_{D^*}, 0_{D^*}) = S(1_{D^*}, 0_{D^*}) = (1, 0, 0) = 1_{D^*},$$
$$\text{and } I(0_{D^*}, 1_{D^*}) = S(1_{D^*}, 1_{D^*}) = (1, 0, 0) = 1_{D^*}.$$

$$I(1_{D^*}, 0_{D^*}) = S(n(1_{D^*}), 0_{D^*}) = S(0_{D^*}, 0_{D^*}) = 0_{D^*},$$
$$I(1_{D^*}, 1_{D^*}) = S(n(1_{D^*}), 1_{D^*}) = S(0_{D^*}, 1_{D^*}) = 1_{D^*}.$$

Moreover, $S(x, y)$ is increasing monotone in each argument, $n(x)$ is non-increasing mapping, then $I(x, y) = S(n(x), y)$ is non-increasing in first argument and is increasing in second one.

Now we give some picture fuzzy implication operators, which are usually referred to in the literature as *S-implications*. ■

Definition 3.10. Let $n(x)$ be a picture negation operator and let $S_{max}(x, y)$ be a picture t-conorm operator. A mapping $I : P^* \times P^* \to D*$ is given by

$$I(x, y) = S_{\max}(n(x), y), \ \forall x, y \in P^*. \tag{2.8}$$

Example 3.1. For $a, b \in P^*$, $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$.

Now we have a new picture implication operator. Since

$$\min(a, b) = (a_1 \wedge b_1, a_2 \wedge b_2, a_3 \vee b_3), \ \max(a, b) = (a_1 \vee b_1, a_2 \wedge b_2, a_3 \wedge b_3),$$
$$\text{and } n_S(a) = (a_3, z_4, a_1), \text{ where } z_4 = 1 - \sqrt{(a_1^2 + a_2^2 + a_3^2)}.$$

We have

$$\begin{aligned} I(a, b) &= S_{\max}(n(a), b) = S_{\max}((a_3, z_4, a_1), (b_1, b_2, b_3)) \\ &= (a_3^2 \vee b_1^2, z_4^2 \wedge b_2^2, a_1^2 \wedge b_3^2), \ \forall a, b \in P^*. \end{aligned} \tag{2.9}$$

If we use $n_0(a) = (a_3, 0, a_1)$, we obtain

$$\begin{aligned} I(a, b) &= S_{\max}(n_0(a), b) = S_{\max}((a_3, 0, a_1), (b_1, b_2, b_3)) \\ &= (a_3^2 \vee b_1^2, 0, a_1^2 \wedge b_3^2), \ \forall a, b \in P^*. \end{aligned} \tag{2.10}$$

Picture fuzzy implication operators on PPFS defined in (2.9) or (2.10) are generalizations of the Kleene-Dienes implication $I_b(x, y) = \max(1 - x, y)$, where $x, y \in [0, 1]$, in the fuzzy logic.

Definition 3.11. A mapping $I : P^* \times P^* \to D*$ is given by

$$I(a,b) = \begin{cases} 1_{D^*} \text{ if } \ a <_1 1_{D^*} \text{ or } b = 1_{D^*}, \\ 0_{D^*} \text{ otherwise,} \end{cases}$$

where $a \in P^*,\ b \in P^*$.

It is a direct generalization of the standard sharp classical implication operator.

Proposition 3.7. *It is a picture implication operator on PPFS of the class 2.*

Proof.

If $a <_1 1_{D^*}$ then $I(a,\ b) = 1_{D^*}\ \forall\ b \in P^*$ then $I(0_{D^*}, b) = 1_{D^*} \Rightarrow I(0_{D^*}, 0_{D^*}) = 1_{D^*}$ and $I(0_{D^*}, 1_{D^*}) = 1_{D^*}$.

It means that the conditions (2.1) and (2.2) are satisfied. The condition (2.3) is satisfied since $b = 1_{D^*} \Rightarrow I(a, b) = 1_{D^*},\ \forall a \in P^*$.

The conditions (2.4) and (2.5) are satisfied from the Definition 3.4.

Definition 3.12. A mapping $I : P^* \times P^* \to D*$ is given by

$$I(a,b) = \begin{cases} 1_{D^*} & \text{if } a \leq_1 b, \\ 0_{D^*} & \text{otherwise,} \end{cases}$$

where $a \in P^*,\ b \in P^*$.

It is a direct generalization of the standard strict implication operator.

Proposition 3.8. *The mapping defined in the Definition 3.12 is a picture implication operator of the class 2.*

The proof is direct from the definition.

Another picture implication operator is the following.

Definition 3.13. A mapping $I : P^* \times P^* \to D*$ is given by

$$r = \begin{cases} 1_{D^*} & \text{if } a \leq_1 b, \\ b & \text{otherwise,} \end{cases}$$

where $r = I(a, b) \in D^*,\ \ a \in P^*,\ b \in P^*$.

It is a new direct generalization of the standard strict implication

Proposition 3.9. *It is a picture implication operator of the class 2.*

The proof is direct from the Definition 3.8.

Definition 3.14. Let $n(a)$ is a picture negation. A mapping $I : P^* \times P^* \to D*$ is given by

$$I(a,b) = \begin{cases} 1_{D^*} & \text{if } a \leq_1 b, \\ \max(n(a), b) & \text{otherwise,} \end{cases}$$

where $\ a \in P^*,\ b \in P^*$.

It is a new direct generalization of the Fodor's fuzzy implication in [7] and is also an picture implication operator on PPFS of the class 2.

4. THE COMPOSITIONAL RULE OF INFERENCE

The compositional rule of inference (see [3]) constitutes an inference rule in approximate reasoning in which it is possible to draw vague conclusions from vague premises. The mathematical pattern of the generalized modus ponens is follows.

Let X and Y be variables taking values in U and V, respectively. Let A, A^* and B be fuzzy subsets of appropriate spaces. From "If X is A then Y is B", and "Y is B^*" can be taken as a logical conclusion.

We can view the conditional statement above as a binary fuzzy relation R, that is, a fuzzy set of $U \times V$, and A^* as a unary fuzzy relation on U. As such, the generalized modus ponens can be examined with a general framework of relations. Firstly, if $f : U \to V$ is a function, then the value $b = f(a)$ may be viewed as the image of the projection of $\{a\}$ into V, that is as the set $\{b \in V : (a, b) \in f\}$.

When f is replaced by a relation R, and A is a subset of U, then the image of the projection of A into V is the set

$$B = \{v \in V : (u, v) \in R \text{ for some } u \in A\}.$$

In terms of indicator functions,

$$B(v) = \bigvee_{u \in U} \{(A \times V)(u, v) \wedge R(u, v)\} = \bigvee_{u \in U} \{A(u) \wedge R(u, v)\}.$$

This can be written as $B = R \circ A$, where $\circ$ is the composition operator of two sets.

When R and A^* are fuzzy subsets of $U \times V$ and V, respectively, the same composition $R \circ A^*$ yields a fuzzy subset of V.

When applying this procedure to the generalized modus ponens schema

IF X is A^* THEN (X, Y) is R, $B^* = R \circ A^*$,

where R is a fuzzy relation on $U \times V$ representing the conditional " If X is A then Y is B".

Thus, if we define $R(u, v) = (A(u) \Rightarrow B(v))$ where $\Rightarrow$ is a fuzzy implication operator and more generally, the special t-norm $T(x, y) = x \wedge y$ can be replaced by an arbitrary fuzzy t-norm operator $T(u, v)$ in the composition operation among relations, leading to the result of the Compositional Rule of Inference (CRI) [3, 6].

$$B^*(v) = \bigvee_{u \in U} \{T((A(u) \Rightarrow B(v)), A(u))\}. \tag{3.2}$$

We can choose concrete t-norm operators and concrete fuzzy implication operators to obtain concrete inference procedures in fuzzy logic.

Compositional rule of inference in picture fuzzy logic on PPFS, PPFL-CRI

Let X and Y be variables assuming values in U and V. Consider Pythagorean picture fuzzy facts X *is* A^* and R is a Pythagorean picture fuzzy relation between U and V, where $A^* \in PPFS(U), R \in PPFR(U \times V)$. The PPFL-CRI allows us to infer the Pythagorean picture fuzzy fact B.

Expressing this under the form of an inference schema, we get

$$\text{If } X \text{ is } A^* \text{ and R } (X, Y) \text{ is } R \text{ then } Y \text{ is } B = R \circ A^*. \tag{3.3}$$

We use a picture fuzzy implication operator $I(a, b)$ to define the picture fuzzy relation R.

Given picture fuzzy sets $A \in PFS(U)$ and $B \in PFS(V)$, we calculate,

$$\begin{aligned}&(x_{1R}(u,v), x_{2R}(u,v), x_{3R}(u,v))\\&= I((x_{1A}(u,v), x_{2A}(u,v), x_{1A}(u,v)), (x_{1B}(u,v), x_{2B}(u,v), x_{3B}(u,v)))\end{aligned} \tag{3.4}$$

for every $(u, v) \in U \times V$.

Thus, we defined the picture fuzzy relation R. Using this definition with the picture fuzzy composition operators of picture fuzzy relations on PPFS given in [7], it is clear that the PPFL- ICR is an extension of the fuzzy-based CRI [5].

Using (3.2) and (3.3) with concrete Pythagorean picture fuzzy t-norms, picture fuzzy implication operators combining with a concrete picture composition operator, (which was given in [8] we obtain the conclusions of the PPFL-ICR.

5. A NUMERICAL EXAMPLE

We give an example of a fuzzy inference in medical diagnose with Pythagorean picture fuzzy set information.

Let U =”cold”, V=”sore throat” with

$$U = \begin{Bmatrix} u_1 & = weakcold & = (0.7, 0.1, 0.3) \\ u_2 & = mediumcold & = (0.6, 0.2, 0.1) \\ u_3 & = verycold & = (0.8, 0.1, 0.4) \end{Bmatrix}, \quad V = \begin{Bmatrix} v_1 & = weak & = (0.35, 0.2, 0.4) \\ v_2 & = medium & = (0.5, 0.2, 0.3) \\ v_3 & = strong & = (0.7, 0.1, 0.3) \end{Bmatrix}.$$

We use a picture fuzzy implication operator $I(u_i, v_j) = S_{\max}(n(u_i), v_j)$, for $i = 1, 2, 3,\ j = 1, 2, 3$ to define the picture fuzzy relation R.

Using picture negation $n(a) = n_0(a)$, we have

$$n(u_1) = n_0(u_1) = (0.3, 0, 0.7),\ n(u_2) = n_0(u_2) = (0.1, 0, 0.6),\ n(u_3) = n_0(u_3) = (0.4, 0, 0.8).$$

The picture fuzzy relation is defined by $R(u_i, v_j) = I(u_i, v_j) = S_{\max}(n_0(u_i), v_j)$, for $i = 1, 2, 3,\ j = 1, 2, 3$. We obtain

$$R(u_1, v_1) = S_{\max}((0.3, 0, 0.7), (0.35, 0.2, 0.4)) = ((0.35)^2, 0, 0.16),$$

$$R(u_1, v_2) = S_{\max}((0.3, 0, 0.7), (0.5, 0.2, 0.3)) = (0.25, 0, 0.09),$$

$$R(u_1, v_3) = S_{\max}((0.3, 0, 0.7), (0.7, 0.1, 0.3)) = (0.49, 0, 0.09).$$

Analogously, we have

$$R(u_2, v_1) = S_{\max}((0.1, 0, 0.6), (0.35, 0.2, 0.4)) = ((0.35)^2, 0, 0.16),$$

$$R(u_2, v_2) = S_{\max}((0.1, 0, 0.6), (0.5, 0.2, 0.3)) = (0.25, 0, 0.09),$$

$$R(u_2, v_3) = S_{\max}((0.1, 0, 0.6), (0.7, 0.1, 0.3)) = (0.49, 0, 0.09),$$

$$R(u_3, v_1) = S_{\max}((0.4, 0, 0.8), (0.35, 0.2, 0.4)) = (0.16, 0, 0.16),$$

$$R(u_3, v_2) = S_{\max}((0.4, 0, 0.8), (0.5, 0.2, 0.3)) = (0.25, 0, 0.09),$$

$$R(u_3, v_3) = S_{\max}((0.4, 0, 0.8), (0.7, 0.1, 0.3)) = (0.49, 0, 0.09).$$

Let the real input be (a real forecast of "cold")

$A^* \in PPFS(U) = \{A^*(u_1) = (0.3, 0, 0.4), A^*(u_2) =, (0.9, 0, 0.2),\ A^*(u_3) = (0.1, 0.05, 0.5)\}$.

Using Compositional Rule of Inference in Picture Fuzzy setting on PPFS, we have the conclusion

$$B^* \in PPFS(V),\ B^* = A^* \circ R = (B^*(v_1), B^*(v_2), B^3(v_3)),$$

with

$$\begin{aligned}
B^*(v_1) &= \max(\min(A^*(u_1), R(u_1, v_1)), \min(A^*(u_2), R(u_2, v_1)), \min(A^*(u_3), R(u_3, v_1))) \\
&= \max(\min((0.3, 0, 0.4), ((0.35)^2, 0, 0.16)), \min((0.9, 0, 0.2), ((0.35)^2, 0, 0.16)), \\
&\quad \min((0.1, 0.05, 0.5), (0.16, 0, 0.16))) \\
&= \max((0.3, 0, 0.4),\ ((0.35)^2, 0, 0.16), (0.1, 0.05, 0.5))) \\
&= (0.3, 0, 0.4).
\end{aligned}$$

$$\begin{aligned}
B^*(v_2) &= \max(\min(A^*(u_1), R(u_1, v_2)), \min(A^*(u_2), R(u_2, v_2)), \min(A^*(u_3), R(u_3, v_2))) \\
&= \max(\min((0.3, 0, 0.4), (0.25, 0, 0.16)), \min((0.9, 0, 0.2), (0.25, 0, 0.09)), \\
&\quad \min((0.1, 0.05, 0.5), (0.25, 0, 0.09))) \\
&= \max((0.25, 0, 0.16),\ (0.25, 0, 0.09), (0.1, 0.05, 0.5))) \\
&= (0.25, 0, 0.16).
\end{aligned}$$

$$\begin{aligned}
B^*(v_3) &= \max(\min(A^*(u_1), R(u_1, v_3)), \min(A^*(u_2), R(u_2, v_3)), \min(A^*(u_3), R(u_3, v_3))) \\
&= \max(\min((0.3, 0, 0.4), (0.49, 0, 0.09)), \min((0.9, 0, 0.2), (0.49, 0, 0.09)), \\
&\quad \min((0.1, 0.05, 0.5), (0.49, 0, 0.09))) \\
&= \max((0.3, 0, 0.4),\ (0.49, 0, 0.09), (0.1, 0.05, 0.5)) \\
&= (0.49, 0, 0.09).
\end{aligned}$$

We obtain the conclusion $B^* \in PFS(V)$,

$B^* = A^* \circ R = (B^*(v_1) = (0.3, 0, 0.4),\ B^*(v_2) = (0.25, 0, 0.16),\ B^3(v_3) = (0.49, 0, 0.09))$.

6. CONCLUSIONS

In this paper we have presented the Pythagorean Picture Fuzzy Set – a combination of the Picture Fuzzy Set and the Yager's Pythagorean Fuzzy Set. After some basic notions, we introduced some main picture logic operators on PPFS, namely the picture negation, picture t-norm on PPFs, picture t-conorm on PPFS and picture implications on PPFS. The new operators are useful in practical computational intelligence problems.

In the future, we will extend the new theory of Pythagorean Picture Fuzzy Sets such as the distance measure between PPFSs, the algebraic structures on PPFSs. They are important tools to measure the analogousness between objects in PPFS and will be developed based on the picture logic operators on PPFS. Furthermore, the theory of Pythagorean Picture Fuzzy Set for decision-making problems will be studied deeply in the next researches

REFERENCES

[1] L.A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, pp. 338–353, 1965.

[2] L.A. Zadeh, "The concept of a linguistic variable and its application to approximate reasoning," *Information Sciences*, vol. 8, pp. 199-249, 1975.

[3] K. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol.20, pp. 87-96, 1986.

[4] K. Atanassov, *On Intuitionistic Fuzzy Sets Theory,* Springer, 2012, Berlin.

[5] T.H. Nguyen and E.Walker, *A first course in fuzzy logic,* Second Edition, Chapman& Hall CRC, 2000, Boca Raton.

[6] E.P. Klement and R.Mesiar, Logical, Algebraic, *Analytic and Probabilistic Aspects of Triangular Norms*, 1st Edition, Elsevier, 2005.

[7] J. Fodor and M. Roubens, *Fuzzy preference modeling and multicriteria decision support,* Kluwer Academic Pub. London, 1994.

[8] B.C. Cuong and V.Krenovitch, "Picture Fuzzy Sets - a new concept for computational intelligence problems," in *The Proceedings of the 3rd World Congress on Information and Communication Technology*, WICT 2013, Hanoi, IEEE CS, 2013, pp 1-6.

[9] B.C. Cuong, Picture fuzzy sets, *Journal Computer science and Cybernetics*, V. 30(4), 419-430, 2014.

[10] B.C. Cuong, V.Kreinovich, R.T.Ngan, A classification of representable t-norms operators for picture fuzzy sets, *Departmental Technical Reports (CS)*, Paper 1047, 2016. http://digitalcommons.utep.edu/cs_techrep/1047

[11] B.C. Cuong, R.T. Ngan, L.B. Long, "Some new de morgan picture operator triples in picture fuzzy logic," *Journal of Computer Science and Cybernetics*, vol. 33, no. 2, pp. 143-164, 2017.

[12] A.A. Yager, "Pythagorean fuzzy subsets," in *The Proceeding of Joint IFSA World Congress and NAFIPS Annual Meeting, Edmonton*, Canada, 2013, pp. 57-61.

[13] A.A. Yager, A.M. Abbasov, "Pythagorean membership grades, complex numbers, and decision making," *Int. J. Intell. Syst.,* vol.28, no. 5, pp. 436-452, 2013.

[14] A.A. Yager, "Pythagorean membership grades in mylticriteria decision making," in *IEEE Trans. Fuzzy Syst.*, vol. 22, no. 4, pp. 958-965, 2014.

[15] D. Molodtsov, "Soft set theory – First results," *Computer* & *Mathematics with Applications*, vol. 37, no. 4-5, pp. 19-31, 1999.

[16] P.K. Maji, R. Bismas, A.R. Roy, "Soft set theory," *Computer* & *Mathematics with Applications*, vol. 45, pp. 555-562, 2003.

[17] B.C. Cuong, "Pythagorean picture fuzzy sets, Part 1 - Basic notions," *Journal of Computer Science and Cybernetics*, vol. 35, no. 4, 2019.

[18] S. Ashraf, S. Abdulallsh, "Spherical aggregation operator and their application in multiattribute group decision-making," *Int. J. Intell. Syst.*, vol. 34, no. 3, pp. 493-523, 2019.

[19] S. Ashraf, S. Abdullah, M. Aslam, M. Qiyas, and M.A. Kutbi, "Spherical fuzzy sets and its representation of spherical fuzzy t-norms and t-conorms," *Journal of Intelligent* & *Fuzzy Systems*, vol. 36, no. 6, pp. 6089-6102, 2019.

[20] S. Ashraf, S. Abdullah, T. Mahmood, F. Ghani, and T. Mahmood, "Spherical fuzzy sets and their applications in multi-attribute decision making problems," *Journal of Intelligent* & *Fuzzy Systems*, vol. 36, no. 3, pp. 2829-2844, 2019.

[21] M. Wu, T. Chu, H. Fan, "Divergence measures of T-spherical fuzzy sets and its application in pattern recognition," *IEEE Access*, vol. 8, pp. 10208-10221, 2019.

[22] P.H. Phong, B.C. Cuong, "Some intuitionistic linguistic aggregation operators," *Journal of Computer Science and Cybernetics*, vol. 30, no.3, pp. 216-226, 2014.

[23] P.H. Phong, B.C. Cuong, "Multi-criteria group decision making with picture linguistic numbers," *VNU Journal of Science: Computer Science and Communication Engineering*, vol.32, no.3, pp. 38-51, 2016.

[24] L.H. Son and P.H. Thong, "Some novel hybrid forecast methods based on picture fuzzy clustering for weather nowcasting from satellite image sequences," *Applied Intelligence*, vol. 46, no. 1, pp. 1-15, 2017.

[25] L.H. Son, P.V. Viet, and P.V. Hai, "Picture inference system: A new fuzzy inference system on picture fuzzy set," *Applied Intelligence*, vol. 46, no. 3, pp. 652-669, 2017.

[26] L.H. Son, "DPFCM: A novel distributed picture fuzzy clustering method on picture fuzzy sets," *Expert Systems with Applications*, vol. 42, no. 1, pp. 51-66, 2015.

[27] P.H. Thong and L.H. Son, "Picture fuzzy clustering: A new computational intelligence method," *Soft Computing*, vol. 20, no. 9, pp. 3544-3562, 2016.

[28] L.H. Son, "Measuring analogousness in picture fuzzy sets: From picture distance measures to picture association measures," *Fuzzy Optimization and Decision Making*, vol. 16, pp. 359-378, 2017.

[29] P.H. Thong, T.A. Tuan, N.T.H. Minh, and L.H. Son, "One solution for proving convergence of picture fuzzy clustering method," in *Information Systems Design and Intelligent Applications*, Springer, Singapore, 2018, pp. 843-852.

[30] P.H. Thong and L.H. Son, "Online picture fuzzy clustering: A new approach for real-time fuzzy clustering on picture fuzzy sets," in *2018 International Conference on Advanced Technologies for Communications (ATC)*, IEEE, 2018, pp. 193-197.

[31] P.T.M. Phuong, P.H. Thong, and L.H. Son, "Theoretical analysis of picture fuzzy clustering: Convergence and property," *Journal of Computer Science and Cybernetics*, vol. 34, no. 1, pp. 17-32, 2018.

[32] W. Xiao, A. Dey, and L.H. Son, "A study on regular picture fuzzy graph with applications in communication networks," *Journal of Intelligent & Fuzzy Systems*, (2020) (Preprint).

[33] R.T. Ngan, S.C. Arnau, M.A. Diaz, J.M.M. Rubio, P. Lopez, F. Andujar, L.H. Son, V.V. Manh, and F. Smarandache, "The picture fuzzy distance measure in controlling network power consumption," *Journal of Fuzzy Extension and Application*, vol. 1, no. 3, pp. 139-158, 2020. Infinite Study.

[34] P.H. Phong, D.T. Hieu, R.T. Ngan, and P.T. Them, "Some compositions of picture fuzzy relations," in *Proceedings of the 7th National Conference on Fundamental and Applied Information Technology Research (FAIR'7),* Thai Nguyen, 2014, pp. 19-20.

[35] B.C. Cuong, R.T. Ngan, and B.D. Hai, "An involutive picture fuzzy negator on picture fuzzy sets and some De Morgan triples," in *2015 Seventh International Conference on Knowledge and Systems Engineering*, 2015, pp. 126-131.

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