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REVISITING SOME FUZZY ALGEBRAIC STRUCTURES

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Abstract. Following our investigations on particular fuzzy algebraic structures, we revisit fuzzy subgroups and fuzzy ideals and introduce some numerical examples. As usual, we associated relations to fuzzy subgroup and fuzzy ideal. Consequently, right and left cosets modulo a fuzzy relation were introduced. This work and the our previous works can be considered as a continuation of investigations initiated by Abu Osman and Antony in the 1980s. Toward our investigation, we have in mind that by introducing these new definitions, the results that we can get should represent a real generalization of classical and commonly known concepts of algebra.

Keywords. Fuzzy relation, fuzzy equivalence relation, fuzzy coset, fuzzy prime ideal.

1. INTRODUCTION

All the work elaborated on the theory of fuzzy sets as well as the various themes that were generated or associated began after the introduction of the foundations of this theory by L. A. Zadeh in 1965 [1]. Fuzzy logic was the first concept that attracted a large number of scientists. Once the foundations of this logic have been established, a number of research directions have emerged. In particular and over the past three decades, many ideas have been developed to transcript classical algebraic structures in the fuzzy sets theory's frame. Among the main issues studied and developed there are the concepts of fuzzy ideals and fuzzy subgroups. In addition, these areas have taken an important and intensive interest in fuzzy mathematical research activity.

More precisely and briefly, the fuzzy groups were introduced by Azriel Rosenfeld in his paper [2]. The concept of ideals of a ring in the fuzzy frame was introduced by Liu in [3] and in [4], he investigated some other questions related to fuzzy ideals and got important new results. The notions of primarily, maximality and radical of fuzzy ideal have been introduced by Malik in [5,6]. Regular and Noetherian rings were characterized in [7] by Mukherjee and Sen. Their study gave rise to an entire characterization of all prime fuzzy ideal of the ring $\mathbb{Z}$. For their contribution to these topics, Kuroaka and Kuroki in [8] and Kumar in [9], studied fuzzy quotient rings and the results obtained have been used by Lee in [10] to characterize fuzzy Artenian and fuzzy Noetherian rings.

The present paper constitutes a logical follow-up to the paper [11]. We not only introduce new definitions of fuzzy relation modulo a fuzzy subgroup (resp. modulo a fuzzy ideal) and their application, but also we continue our investigation started in [12]. We got many results that can be considered as a generalization of the same results obtained in crisp frame. We

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start by recalling some fuzzy basic concepts for the sake of completeness. We point out that our introduced definitions are more realistic than the classical ones. We have illustrated our results with various numerical examples.

2. PRELIMINARIES

2.1. The indispensable in fuzzy set theory

This section is devoted to recalling certain classical definitions related to the theory of fuzzy sets and to certain well-known properties of these sets. All the concepts presented in this subsection can be found in any document related to this topic (see [13–16]), however [1] can be seen as the keystone of this theory.

Definition 1. [1] Any mapping $\gamma\colon X \to [0,1]$ is called fuzzy set on the universe X. The mapping γ is also called the membership function. The real number $\gamma(x)$ is the grade of membership of x to X.

For a finite set $X = \{x_1, \ldots, x_n\}$, the fuzzy set (X, γ) is often denoted by $\{\gamma(x_1), \ldots, \gamma(x_n)\}$.

Definition 2. [12] Let α be a fixed real number in $[0,1]$, the mapping $\gamma\colon X \to [0,1]$, satisfying $\gamma(x) = \alpha$ for all $x \in X$ is called constant fuzzy set and will be denoted in the sequel by $\underline{\alpha}$. When $\alpha = 0$, γ is the empty fuzzy set and when $\alpha = 1$, γ is the whole set X.

2.2. Fuzzy operations

Definition 3. [12] A mapping $* : X \times X \times X \to [0,1]$ is a fuzzy binary operation on a set X if $\forall a, b \in X$, there exists a unique element $c \in X$ satisfying $*(a,b,c) = 1$.

Let $*$ be a fuzzy binary operation on a set X, then

1. $*$ is commutative if for all $a, b, c \in X$, $*(a,b,c) = *(b,a,c)$.
2. $*$ is associative if $\forall x, y, z, a, b \in X$, $*(x,y,a) > 0$, $*(y,z,b) > 0$ and $*(a,z,\alpha) = *(x,b,\beta)$ imply $\alpha = \beta$ for any α, $\beta \in X$.
3. An element e of X is an identity for $*$ if $*(x,e,x) = *(e,x,x)$, $\forall x \in X$.

Proposition 4. [12] *If e is an identity of a set X for a fuzzy binary operation $*$ and if $*(x,e,x) = 1$, $\forall x \in X$, then e is unique.*

Proof. Suppose that e, e' are two identities of X for $*$ then $1 = *(e', e, e') = f(e, e', e')$ and $1 = *(e, e', e) = *(e', e, e)$ so $*(e', e, e') = *(e', e, e) = 1$. But there exists a unique element x such that $*(e', e, x) = 1$ and then $e = e'$. ■

Definition 5. [12] Let $* : E \times E \times E \to [0,1]$ be a fuzzy operation on a set E and e be the unique identity of E for $*$. An element $x \in E$ is symmetrizable if $*(x, x', e) = *(x', x, e)$ for some $x' \in E$.

Definition 6. [12] Let $* : E \times E \times E \to [0,1]$ be a fuzzy operation on a set E and e be the unique identity of E for $*$. An element $x' \in E$ if it exists such that $*(x, x', e) = *(x', x, e) = 1$ is called a symmetric element of $x \in E$.

Definition 7. [12] Let $* : E \times E \times E \to [0,1]$ be a fuzzy operation on a set E. An element $a \in E$ is left (resp. right) regular or cancelable if for any elements $x, y, z \in E$, the equality

$*(a,x,z) = *(a,y,z)$ (resp. $*(x,a,z) = *(y,a,z)$) implies $x = y$. It is regular or cancelable if it is left and right regular.

Proposition 8. [12] *If a fuzzy operation $*$ on E posses an identity e. Any left (resp. right) regular element has at most one symmetric element.*

Proof. Suppose that an element x is left regular and has two symmetric x' and x''. We then have $*(x,x',e) = f(x',x,e) = *(x,x'',e) = *(x'',x,e) = 1$ and consequently $*(x,x',e) = *(x,x'',e) = 1$. The last equality implies that $x' = x''$. For the right regularity the proof is trivial. ∎

2.3. Revisiting fuzzy subgroups and fuzzy ideals

To make the present paper more readable, we start by recalling the results obtained for these topics in the papers [12] and [11]. The groups are not necessary commutative but rings considered are supposed to be commutative and unitary if no restriction is specified.

Definition 9. [11, 12] Let $(G,\star)$ be a group, e be its identity. A fuzzy subset γ of G is a fuzzy subgroup of G if and only if

1. $\gamma(e) = 1$,
2. $\forall a,b \in G,\ \gamma(a\star b) \geq \min\{\gamma(a),\gamma(b)\}$,
3. $\forall a \in G;\ \gamma(a) = \gamma(a^{-1})$.

It is said normal if in addition $\gamma(a\star b) = \gamma(b\star a),\ \forall a,b \in G$.

Proposition 10. [11] *γ is a normal fuzzy subgroup of the group G if and only if*

$$\gamma(a\star b\star a^{-1}) = \gamma(b),\ \forall a,b \in G.$$

Proof. If $\gamma(a\star b) = \gamma(b\star a), \forall a,b \in G$, then

$$\gamma(a\star b\star a^{-1}) = \gamma(a^{-1}\star(a\star b)) = \gamma((a^{-1}\star a)\star b) = \gamma(e\star b) = \gamma(b),\ \forall a,b \in G.$$

Conversely, if $\forall a,b \in G,\ \gamma(a\star b\star a^{-1}) = \gamma(b)$, then

$$\gamma(a\star b) = \gamma(a\star b\star a\star a^{-1}) = \gamma(a\star(b\star a)\star a^{-1}) = \gamma(b\star a),\ \forall a,b \in G.$$

∎

Definition 11. [11] (Classical definition) Let $(R,+,\times)$ be a ring and 0_R its identity for $+$, a fuzzy subset γ of R is called a fuzzy ideal of R if

1. $\gamma(0_R) = 1$,
2. $\gamma(a) = \gamma(-a)$,
3. $\gamma(a+b) \geq \min\{\gamma(a),\gamma(b)\},\ \forall a,b \in R$,
4. $\gamma(a\times b)\ \geq \max\{\gamma(a),\gamma(b)\},\ \forall a,b \in R$.

By changing the last condition in the above definition we get another definition of a fuzzy ideal of a ring. It will be as follows.

Definition 12. [11] (New definition) Let $(R, +, \times)$ be a ring and 0_R its identity for $+$, a fuzzy subset γ of R is a right (respectively left) fuzzy ideal of R if

1. $\gamma(0_R) = 1$,
2. $\gamma(a) = \gamma(-a)$,
3. $\gamma(a + b) \geq \min\{\gamma(a), \gamma(b)\},\ \forall a, b \in R$,
4. $\gamma(a) > 0 \rightarrow \gamma(a \times b) > 0$ (respectively $\gamma(b \times a) > 0$), $\forall a, b \in R$.

It is said proper if $\gamma(1) \neq 1$.

Remark 1.

1. It is easy to see that the second definition is less restrictive than the first one. Moreover, if γ verifies the axioms of the first definition, it verifies the axioms of the second definition. The converse is false.
2. Since if we assume that γ is an ideal such that $\gamma(1) \geq a \in [0, 1]$, we have necessary $\gamma(x) = \gamma(x.1) \geq \sup\{\gamma(1), \gamma(x)\} \geq a,\ \forall x \in X$. Thus, if $a = 1$ then γ becomes the constant fuzzy set $\underline{1}$ (the whole ring). Consequently, the terminology of proper ideal means that $\gamma(1) \neq 1$.
3. The membership function γ indicates the behavior of the elements of the ring with respect to the ideal, it is trivial that in the definition of fuzzy ideal commonly used, the product $a.b$ is closer to the ideal than each of the elements a and b. In counter part in our introduced definition the element $a.b$ is close to the ideal (i.e. $\gamma(a.b) > 0$) if one of the elements a or b is close to the ideal. As one can notice that this definition is more realistic and coincides with the definition of crisp ideal when γ takes its values in $\{0, 1\}$.

3. MAIN RESULTS

Axioms 2 and 3 in the above definition can be combined in only one axiom and we get.

Proposition 13. *A fuzzy ideal is a fuzzy subset γ of a ring $(R, +, .)$ that verifies*

1. $\gamma(0_R) = 1$,
2. $\gamma(a - b) \geq \min\{\gamma(a), \gamma(b)\},\ \forall a, b \in R$,
3. $\gamma(a) > 0 \rightarrow \gamma(a \times b) = \gamma(b \times a) > 0,\ \forall a, b \in R$.

Proof. It is clear that Definition 17 implies the 3 axioms of the proposition. Conversely, from second axiom of the proposition, we have

1. For any $a \in R$, $\gamma(a) = \gamma(0 - (-a)) \geq \min\{\gamma(0), \gamma(-a)\} = \gamma(-a)$.

2. For any $a \in R$, $\gamma(-a) = \gamma(0 - a) \geq \min\{\gamma(0), \gamma(a)\} = \gamma(a)$.

So $\gamma(a) = \gamma(-a)$.

On the other hand $\gamma(a + b) = \gamma(a - (-b)) \geq \min\{\gamma(a), \gamma(-b)\} = \min\{\gamma(a), \gamma(b)\}$. ■

Remark 2. If $(R, +, .)$ is a commutative ring, any right (resp. left) fuzzy ideal is a fuzzy ideal. As we will be only concerned in the sequel by such rings, the meaning of fuzzy ideal is according to the previous definition.

Proposition 14. *If γ is a proper fuzzy ideal of a ring R, then $\gamma(a) = 0$ for any invertible element $a \in R$.*

Proof. The proof is straightforward. ■

Definition 15. A fuzzy ideal is said prime if $\gamma \neq \underline{1}$, $\gamma(a \times b) > 0 \to \gamma(a) > 0$ or $\gamma(b) > 0$, $\forall a, b \in R$.

It is clear, since $\gamma \neq \underline{1}$ that $\gamma(1) = 0$.

The following proposition characterizes some type of fuzzy prime ideals.

Proposition 16. *Let $\alpha \in]0, 1[$. If γ is a fuzzy ideal of an integral domain R satisfying,*

$$\gamma(a) = \begin{cases} 1 & \text{if } a = 0 \\ 0 & \text{if } a \textit{ is invertible} \\ \alpha & \textit{otherwise}, \end{cases}$$

then γ is fuzzy prime ideal.

Proof. Suppose that $a, b \in R$ are such $\gamma(ab) > 0$ so ab is not invertible.

1. If $ab = 0$ then, since R is an integral domain, $a = 0$ or $b = 0$ and consequently $\gamma(a) > 0$ or $\gamma(b) > 0$.

2. Assume now that $ab \neq 0$. We have to distinguish three cases:

   (a) If $a \neq 1$ and a is not invertible, then $\gamma(a) > 0$.

   (b) If $a = 1$, then $\gamma(b) = \gamma(1b) = \gamma(ab) > 0$ and the result follows.

   (c) If a is invertible, then there exists $c \in R$ such that $ca = 1$. On the other hand since γ is an ideal, $\gamma(ab) > 0 \to \gamma(c(ab)) > 0$. But $0 < \gamma(c(ab)) = \gamma((ca)b) = \gamma(1b) = \gamma(b)$ and the result follows.

■

Let us consider the following example.

Example 17. Consider the fuzzy ideal γ on the ring $(\mathbb{Z}/8\mathbb{Z}, +, .)$ given by

a	0	1	2	3	4	5	6	7
$\gamma(a)$	1	0	0.5	0	0.5	0	0.5	0

From the above proposition, γ is fuzzy prime ideal. Moreover, one can check that γ is not maximal. We can also prove that γ is prime directly. Indeed, let us draw the tables corresponding to $\gamma(a + b)$ and $\gamma(ab)$ for different values of a and b in $\mathbb{Z}/8\mathbb{Z}$.

Table 1. Values of $\gamma(a+b),\ a,b \in \mathbb{Z}/8\mathbb{Z}$

$a\diagdown b$	0	1	2	3	4	5	6	7
0	1	0	0.5	0	0.5	0	0.5	0
1	0	0.5	0	0.5	0	0.5	0	1
2	0.5	0	0.5	0	0.5	0	1	0
3	0	0.5	0	0.5	0	1	0	0.5
4	0.5	0	0.5	0	1	0	0.5	0
5	0	0.5	0	1	0	0.5	0	0.5
6	0.5	0	1	0	0.5	0	0.5	0
7	0	1	0	0.5	0	0.5	0	0.5

Table 2. Values of $\gamma(a.b),\ a,b \in \mathbb{Z}/8\mathbb{Z}$

$a\diagdown b$	0	1	2	3	4	5	6	7
0	1	1	1	1	1	1	1	1
1	1	0	0.5	0	0.5	0	0.5	0
2	1	0.5	0.5	0.5	1	0.5	0.5	0.5
3	1	0	0.5	0	0.5	0	0.5	0
4	1	0.5	1	0.5	1	0.5	1	0.5
5	1	0	0.5	0	0.5	0	0.5	0
6	1	0.5	0.5	0.5	1	0.5	0.5	0.5
7	1	0	0.5	0	0.5	0	0.5	0

It is clear that γ is a fuzzy ideal. Moreover $\gamma(a.b) > 0$ implies $\gamma(a)\ > 0$ or $\gamma(b)\ > 0$. So γ is a fuzzy prime ideal.

Now if define β to be the following fuzzy ideal of $\mathbb{Z}/8\mathbb{Z}$

a	0	1	2	3	4	5	6	7
$\beta(a)$	1	0	0.7	0	0.7	0	0.7	0

we have $\gamma < \beta$ and $\beta \neq \underline{1}$. So γ is not maximal according to Definition 21 [17].

Example 18. [Counter example] Let β be the following fuzzy set of $\mathbb{Z}/8\mathbb{Z}$.

a	0	1	2	3	4	5	6	7
$\gamma(a)$	1	0	0.2	0	0.4	0	0.7	0

Since $0.4 = \beta(2 \times 6) < 0.7 = \max\{\beta(2), \beta(6)\}$, then β is not a fuzzy ideal of $\mathbb{Z}/8\mathbb{Z}$.

Next, we recall again some results obtained in [12], improve some others, and introduce new results and examples.

Proposition 19. [12] *If γ is a proper fuzzy ideal of R then for all $\theta \in [0,1]$ the set $I = \{x \in R \mid \gamma(x) \geq \theta\}$ is an ideal of the ring $(R, +, \times)$.*

If in addition, if $\forall a, b \in R,\ a \times b \in I$ implies $\gamma(a \times b) = \gamma(a).\gamma(b)$ then I is a prime ideal of the ring R.

Proposition 20. [12] *Let $(R, +, \times)$ be a ring and 0_R be its identity for $+$, the intersection $\alpha \wedge \gamma$ of two fuzzy ideals α and γ of R is a fuzzy ideal of R.*

In the sequel, if α is a fuzzy set of R, we define for any $n \in \mathbb{N}$, α^n by $\forall x \in R$, $\alpha^n(x) = (\alpha(x))^n$.

With a view to improving the definition of fuzzy radical introduced in [11, 12], we give the following proposition.

Proposition 21. *Let γ be a fuzzy ideal (in classical sense) of a ring R and let $\beta : R \longrightarrow [0,1]$ defined by $\beta(x) = \sup_{n\in\mathbb{N}}\{\gamma(x^n)\}$ such that $\beta(a,b) \geq \min\{\beta(a), \beta(b)\}$. Then,*

1. *β is an ideal,*
2. *If γ is prime then β is also prime,*
3. *If γ is maximal, then $\beta = \gamma$.*

Proof

1. (a) $\beta(0) = \sup_{n\in\mathbb{N}}\{\gamma(0^n)\} = \gamma(0) = 1.$

   (b) $\beta(-x) = \sup_{n\in\mathbb{N}}\{\gamma((-x)^n)\} = \sup_{n\in\mathbb{N}}\{\gamma((-1)^n x^n)\} \sup_{n\in\mathbb{N}}\{\gamma(x^n)\} = \beta(x)$ since $\gamma((-1)^n x^n) = \gamma(x^n)$, $\forall n \in \mathbb{N}$.

   (c) Assume that $\beta(x) > 0$ and let $y \in R$, then $\beta(xy) = \sup_{n\in\mathbb{N}}\{\gamma((xy)^n)\} = \sup_{n\in\mathbb{N}}\{\gamma(x^n y^n)\} \geq \sup_{n\in\mathbb{N}}(\max\{\gamma(x^n),\ \gamma(y^n)\}) > 0$, otherwise $\gamma(x^n) = 0$, $\forall n \in \mathbb{N}$ and then $\sup_{n\in\mathbb{N}}(\max\{\gamma(x^n)\} = 0$. This is a contradiction.

2. We first notice that $\beta(1) = \sup_{n\in\mathbb{N}}\{\gamma(1^n)\} = \gamma(1) = 0$, so $\beta \neq \underline{1}$.

   Let $x, y \in R$ such that $\beta(xy) > 0$, so $\sup_{n\in\mathbb{N}}\{\gamma((xy)^n)\} = \sup_{n\in\mathbb{N}}\{\gamma(x^n y^n)\} > 0$ and then there exists m_0 such that $\gamma(x^{m_0} y^{m_0}) > 0$ and since γ is prime, we obtain, $\gamma(x^{m_0}) > 0$ or $\gamma(y^{m_0}) > 0$ and consequently $\sup_{n\in\mathbb{N}}\{\gamma(x^n)\}$ or $\sup_{n\in\mathbb{N}}\{\gamma(y^n)\} > 0$ and finally $\beta(x) > 0$ or $\beta(y) > 0$ and β is prime.

3. Notice that $\forall x \in R$, $\gamma(x) \leq \beta(x)$. Since γ is maximal $\beta = \gamma$ or $\beta = \underline{1}$, but we have proved that $\beta \neq \underline{1}$ and the conclusion follows.

■

Example 22. Let γ be the fuzzy ideal given by Example 17, if β is the fuzzy set given by $\beta(x) = \sup_{n\in\mathbb{N}}(\gamma(x^n))$, then we have $\beta(a.b) \geq \min\{\beta(a), \beta(b)\}$. By Proposition 21, β is not only a fuzzy ideal but also prime one. Moreover it is simple to check that $\beta = \gamma$ but as we have said previously, γ is not maximal and then the converse of the third assertion in Proposition 21 is not true.

In Proposition 36 of [12] , we supposed that a must be in the center $Z(R)$ of the ring R. Next, we show that this condition is not necessary.

Proposition 23. *Let γ be a fuzzy ideal on a ring R and a be an element of R.*

1. *The fuzzy set $\gamma_a : R \to [0,1]$ defined by $\gamma_a(x) = \gamma(a.x)$ is a fuzzy ideal of R.*

2. *If γ is prime, then γ_a is prime.*

3. *If γ is maximal, the ideal γ_a is equal to γ or to $\underline{1}$. Indeed if there exists $x \in R$ such that $\gamma_a(x) = 1$ then $\forall x \in R,\ \gamma_a(x) = \gamma(x)$.*

Proof

1. (a) $\gamma_a(0) = \gamma(a.0) = \gamma(0) = 1$,

   (b) $\gamma_a(-x) = \gamma(a(-x)) = \gamma(-(ax)) = \gamma(ax) = \gamma_a(x)$,

   (c) $\gamma_a(x+y) = \gamma(a(x+y)) = \gamma(ax+ay) \geq \min\{\gamma(ax), \gamma(ay)\} = \min\{\gamma_a(x), \gamma_a(y)\}$,

   (d) Assume that $\gamma_a(x) > 0$, then for $y \in R$ we have $\gamma_a(xy) = \gamma(a(xy)) = \gamma((ax)y)) \geq \max\{\gamma(ax), \gamma(y)\} \geq \gamma(ax) = \gamma_a(x) > 0$, so the result follows.

2. Suppose that $x, y \in R$ are such that $\gamma_a(xy) > 0$, then $\gamma((ax)y) = \gamma(a(xy)) > 0$ and since γ is prime, we get $\gamma(ax) > 0$ or $\gamma(y) > 0$. If $\gamma(ax) = 0$, as $\gamma(y) > 0$ and $\gamma(ay) \geq \max\{\gamma(a), \gamma(y)\} > 0$ and then γ_a is prime.

3. Notice that for all $x \in R$, $\gamma_a(x) \geq \max\{\gamma(a), \gamma(x)\} \geq \gamma(x)$. Since γ is maximal, we have either $\gamma_a = \underline{1}$ or $\gamma_a = \gamma$. Now suppose that $\gamma_a \neq \underline{1}$, then $\gamma_a = \gamma$.

■

Example 24. Let γ be the fuzzy ideal given by Example 17. Define the fuzzy sets $\gamma_{i\in\{2,3,4\}}$, by

$$\gamma_i : \mathbb{Z}/8\mathbb{Z} \to\ [0,1],\ x \mapsto\ \gamma_i(x) = \gamma(i \times x).$$

The corresponding tables of the fuzzy sets are

a	0	1	2	3	4	5	6	7
$\gamma_2(a)$	1	0.5	0.5	0.5	1	0.5	0.5	0.5
$\gamma_3(a)$	1	0	0.5	0	0.5	0	0.5	0
$\gamma_4(a)$	1	0.5	1	0.5	1	0.5	0	0.5

By Proposition 23, for $i \in \{2,3,4\}$, γ_i is a prime ideal. As for $i \in \{2,3,4\}$, γ_i is neither equal to γ nor to $\underline{1}$, γ is not maximal, which confirms the non-maximality of γ as proved in Example 17. Moreover, it is simple to check that $\gamma_3 \leq \gamma_2$. The prime ideals γ_4 and γ_2 or the prime ideals γ_4 and γ_3 are incomparable.

Definition 25. Let a be an element of the commutative ring $(R,+,.)$ and γ be a fuzzy ideal of R.

1. γ is invariant by the homothecy $\mathfrak{h}_a$ if $\mathfrak{h}_a(\gamma) = \gamma$, where $\mathfrak{h}_a(\gamma) = \gamma_a$.

2. γ is invariant by the translation τ_a if $\tau_a(\gamma) = \gamma$, where $\tau_a(\gamma)(x) = \gamma(a+x)$.

Proposition 26. *Let a be an element of the commutative ring $(R,+,.)$ and γ be a fuzzy ideal of R.*

1. *If γ is invariant by the translation τ_a then $\gamma_a = \underline{1}$.*

2. *If γ is invariant by the homethecy $\mathfrak{h}_a$ then $\gamma(a) = 0$.*

Proof

1. If γ is invariant by the translation τ_a, then $\gamma(a+x) = \gamma(x)$ for all $x \in R$ so
$$\gamma(a) = \gamma(a+0) = \gamma(0) = 1.$$
On the other hand, $1 \geq \gamma_a(x) = \gamma(ax) \geq \max\{\gamma(a), \gamma(x)\} = \gamma(a) = 1$ and then $\gamma_a = \underline{1}$.

2. If γ is invariant by the homethecy $\mathfrak{h}_a$ then $\gamma(ax) = \gamma(x)$, $\forall x \in R$, so $\gamma(a) \leq \max\{\gamma(a), \gamma(x)\} \leq \gamma(ax) = \gamma(x)$, $\forall x \in R$. Consequently, for $x = 1$, we get $\gamma(a) \leq \gamma(1) = 0$. The result follows. ■

Example 27. Let $r_x \in \{0, 1, 2\}$ be the remainder of the euclidian division (long division) of x by 3 in $\mathbb{Z}$. If for any element $x \in \mathbb{Z}$, we set

$$\gamma(x) = \begin{cases} 1 & \text{if } r_x = 0 \\ 1/4 & \text{if } r_x \neq 0, \end{cases}$$

γ is then a fuzzy ideal of $\mathbb{Z}$ in the sense of Definition 16.

First, let us give explicitly the values taken by $\gamma(x+y)$ and $\gamma(xy)$ for different values of r_x and r_y.

Table 3. Values of $\gamma(x+y)$, $r_x, r_y \in \{0, 1, 2\}$

$r_x \diagdown r_y$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$\xrightarrow{\gamma}$

$\gamma(r_x) \diagdown \gamma(r_y)$	1	1/4	1/4
1	1	1/4	1/4
1/4	1/4	1/4	1
1/4	1/4	1	1/4

Table 4. Values of $\gamma(xy)$, $r_x, r_y \in \{0, 1, 2\}$

$r_x \diagdown r_y$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

$\xrightarrow{\gamma}$

$\gamma(r_x) \diagdown \gamma(r_y)$	1	1/4	1/4
1	1	1	1
1/4	1	1/4	1/4
1/4	1	1/4	1/4

We can prove directly that γ is a proper fuzzy ideal of $(\mathbb{Z}, +, .)$.

1. By definition and since $0 \equiv 0[3]$, then $\gamma(0) = 1$.

2. From the equivalence $x \equiv 0[3] \iff -x \equiv 0[3]$, we deduce that $\gamma(x) = \gamma(-x)$.

3. From Table 3, it is easy to see that $\gamma(x+y) \geq \min\{\gamma(x), \gamma(y)\}$, $\forall x, y \in \mathbb{Z}$.

4. The inequality $\gamma(xy) \geq \max\{\gamma(x), \gamma(y)\}$, $\forall x, y \in \mathbb{Z}$ can be deduced from Table 4.

The proof of Proposition 47 of [12] contains a mistake. Next, we recall the proposition and give the adequate proof.

Proposition 28. [12] *Let $\theta \in]0, 1]$ be fixed, a be an element of a commutative ring $(R, +, .)$ and $\gamma : R \to [0, 1]$ be a mapping satisfying the following conditions.*

1. $\theta \geq \gamma(a) \geq \dfrac{\theta}{2}$,

2. $\gamma(0) = 1$,

3. $\gamma(x) = \begin{cases} \gamma(a) & \textit{if} \quad x = a.y \quad \textit{for some} \quad y \in R \setminus \{0\} \\ \theta - \gamma(a) & \textit{otherwise.} \end{cases}$

Then γ is an ideal. ■

Proof. Notice that since $\gamma(a) \geq \dfrac{\theta}{2}$, we have $\gamma(a) \geq \theta - \gamma(a)$. Let us draw the tables giving $\gamma(x+y)$ and $\gamma(xy)$ for different values of x and y. Next, $x \notin \langle a \rangle$ means that x cannot be written as a product of a and an element of R.

Table 5. Different values of $x + y$ and $\gamma(x + y)$, for $x, y \in R$

$x \diagdown y$	0	ay'	$\notin \langle a \rangle$
0	0	ay'	$\notin \langle a \rangle$
ax'	ax'	az	$\notin \langle a \rangle$
$\notin \langle a \rangle$	$\notin \langle a \rangle$	$\notin \langle a \rangle$	?

$\overrightarrow{\gamma}$

$\gamma(x) \diagdown \gamma(y)$	1	$\gamma(a)$	$\theta - \gamma(a)$
1	1	$\gamma(a)$	$\theta - \gamma(a)$
$\gamma(a)$	$\gamma(a)$	$\gamma(a)$	$\theta - \gamma(a)$
$\theta - \gamma(a)$	$\theta - \gamma(a)$	$\theta - \gamma(a)$	α

Table 6. Different values of xy and $\gamma(xy)$, for $x, y \in R$

$x \diagdown y$	0	ay'	$\notin \langle a \rangle$
0	0	0	0
ax'	0	az	az
$\notin \langle a \rangle$	0	az	?

$\overrightarrow{\gamma}$

$\gamma(x) \diagdown \gamma(y)$	1	$\gamma(a)$	$\theta - \gamma(a)$
1	1	1	1
$\gamma(a)$	1	$\gamma(a)$	$\gamma(a)$
$\theta - \gamma(a)$	1	$\gamma(a)$	α'

Since $\max\{\theta - \gamma(a), \theta - \gamma(a)\} = \min\{\theta - \gamma(a), \theta - \gamma(a)\}$, then $\alpha \geq \min\{\theta - \gamma(a), \theta - \gamma(a)\}$ and $\alpha' \geq \max\{\theta - \gamma(a), \theta - \gamma(a)\}$, so the axioms $\gamma(x + y) \geq \min\{\gamma(x), \gamma(y)\}$ and $\gamma(xy) \geq \max\{\gamma(x), \gamma(y)\}$ are verified for all the possible values taken by $\gamma(x + y)$ and $\gamma(xy)$ in the last case in the tables. On the other hand, it is easy to show that $\gamma(-x) = \gamma(x)$. ■

Remark 3. If $\gamma(a) = \theta = 1$, the fuzzy ideal γ coincides with the principal ideal generated by a in the ring R.

As an illustration of the above proposition, let us investigate an example on the ring $\mathbb{Z}$.

Example 29. Let γ_2, γ_3, γ_6, $\theta \in [0, 1]$ such that $\gamma_6 \geq \gamma_2 \vee \gamma_3$, $2\gamma_6 \leq \theta$. We define for $n \in \{2, 3\}$

$$\gamma_n(x) = \begin{cases} 1 & \text{if } x = 0 \\ \gamma_n & \text{if } x \in n\mathbb{Z} \setminus \{0\} \\ \theta - \gamma_n & \text{otherwise,} \end{cases}$$

$$\gamma_6(x) = \begin{cases} 1 & \text{if } x = 0 \\ \gamma_6 & \text{if } x \in 6\mathbb{Z} \setminus \{0\} \\ \theta - \gamma_6 & \text{if } x \text{ either in } 2\mathbb{Z} \setminus 3\mathbb{Z} \text{ or in } 3\mathbb{Z} \setminus 2\mathbb{Z} \\ \beta & \text{otherwise.} \end{cases}$$

If $\beta \geq \max\{\theta - \gamma_2, \theta - \gamma_3\}$, then $\gamma_6 \subset \gamma_2 \wedge \gamma_3$.

1. If $x \in 6\mathbb{Z}$, then $\gamma_6(x) = \gamma_6$. But in this case, we also have $x \in 2\mathbb{Z}$ and $x \in 3\mathbb{Z}$, so $\gamma_2(x) = \gamma_2$ and $\gamma_3(x) = \gamma_3$ and from our conditions, we get

$$\gamma_6(x) = \gamma_6 \geq \max\{\gamma_2, \gamma_3\} = \max\{\gamma_2(x), \gamma_3(x)\} \geq \min\{\gamma_2(x), \gamma_3(x)\}.$$

2. If $x \notin 6\mathbb{Z}$ and $x \in 3\mathbb{Z}$, then $\gamma_6(x) = \theta - \gamma_6,\ \gamma_2(x) = \theta - \gamma_2,\ \gamma_3(x) = \gamma_3$. From $2\gamma_6 \leq \theta$, we deduce that $\gamma_3 \leq \gamma_6 \leq \theta - \gamma_6$.

   On the other hand, $\gamma_2 \leq \gamma_6 \rightarrow \theta - \gamma_6 \leq \theta - \gamma_2$ and then $\min\{\gamma_3, \theta - \gamma_2\} = \gamma_3$. Consequently,
$$\gamma_6(x) = \theta - \gamma_6 \geq \gamma_3 = \min\{\gamma_3(x), \gamma_2(x)\}.$$

3. The case where $x \notin 6\mathbb{Z}$ and $x \in 2\mathbb{Z}$ can be proved by the same arguments as in 2).

4. Let $x \notin (2\mathbb{Z} \cup 3\mathbb{Z})$. This case is trivial since

$$\gamma_6(x) = \theta - \gamma_6 = \beta \geq \min\{\theta - \gamma_2, \theta - \gamma_2\} = \min\{\gamma_2(x), \gamma_3(x)\}.$$

3.1. Relation associated to a fuzzy subgroup

Definition 30. [12] A fuzzy relation R on a set X is a mapping $R : X \times X \longrightarrow [0,\ 1]$.

Definition 31. [12] A fuzzy relation R on a set X is:

1. Reflexive if $\forall x \in X, R(x, x) = 1$.

2. Symmetric if $\forall x,\ y \in X,\ R(x, y) = R(y, x)$,

3. Transitive if $\forall x,\ y,\ z \in X, R(x, y) > 0$ and $R(y, z) > 0$ imply $R(x, z) > 0$ or equivalently $R(x, z) \geq \max\{R(x, y),\ R(y, z)\}$.

4. A fuzzy relation R on a set X is an equivalence relation on X if it is reflexive, symmetric and transitive.

Proposition 32. *Let γ be a fuzzy subgroup of a group G. The fuzzy relation $R : G \times G \longrightarrow [0,\ 1]$ denoted by R_γ and defined by*

$$\forall x,\ y \in G,\ R_\gamma(x, y) = \gamma(xy^{-1})$$

is an equivalence relation on G called the right fuzzy relation modulo the fuzzy subgroup γ.

Proof

1. Reflexivity: if $\forall x \in X, R_\gamma(x, x) = \gamma(xx^{-1}) = \gamma(e) = 1$.

2. Symmetry: assume that $x,\ y \in X$, then

$$R_\gamma(x, y) = \gamma(xy^{-1}) = \gamma((xy^{-1})^{-1}) = \gamma(yx^{-1}) = R_\gamma(y, x).$$

3. Transitivity: let $x,\ y,\ z$ be elements of X, such that $R_\gamma(x, y) > 0$ and $R_\gamma(y, z) > 0$.

$$\begin{aligned} R_\gamma(x,z) &= \gamma(xz^{-1}) \\ &= \gamma((xy^{-1})(yz^{-1})) \geq \max\{\gamma(xy^{-1}), \gamma(yz^{-1})) = \max\{R_\gamma(x,y),\ R_\gamma(y,z)\} > 0 \end{aligned}$$

so $R_\gamma(x,z) > 0$. ■

Example 33. Let G be the Klein four-group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, the fuzzy set γ given by

$$\gamma(0,0) = 1,\ \gamma(0,1) = 0.7,\ \gamma(1,0) = 0.5,\ \gamma(1,1) = 0.5$$

is a fuzzy subgroup of G. If we set $\forall x,\ y \in G,\ R_\gamma(x,y) = \gamma(x-y)$, we have $R_\gamma(x,y) = \gamma(x+y)$ and the table of its values will be given by Table 7.

Table 7. Values of $R_\gamma(a,b) = \gamma(a-b),\ a,b \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

$a\diagdown b$	(0,0)	(0,1)	(1,0)	(1,1)
(0,0)	1	0.7	0.5	0.5
(0,1)	0.7	1	0.5	0.5
(1,0)	0.5	0.5	1	0.7
(1,1)	0.5	0.5	0.7	1

Definition 34. Let γ be a fuzzy subgroup of a group G and R_γ be the relation associated to γ by the previous proposition. For any $x \in G$, the fuzzy set

$$\gamma_x : G \longrightarrow [0,1],\ \gamma_x(y) = R_\gamma(x,y) = \gamma(xy^{-1})$$

is called the right coset modulo R_γ of x.

Notice that $\gamma_e = \gamma$.

Proposition 35. *Let γ be a fuzzy subgroup of a group G and R_γ be the relation associated to γ. The set of right cosets $G/R_\gamma = \{\gamma_x,\ x \in X\}$, modulo R_γ forms a partition of G, that is, it satisfies:*

1. *For any $x \in G,\ \gamma_x \neq \underline{0}$,*
2. *If for $x,y \in G,\ \gamma_x(y) = 0$ then $\gamma_x \cap \gamma_y = \underline{0}$,*
3. *If for $x,y \in G,\ \gamma_x(y) > 0$ then $Supp(\gamma_x) = Supp(\gamma_y)$,*
4. $\bigcup_{x\in G} \gamma_x = \underline{1}$.

Proof

1. For any $x \in G$ we have $\gamma_x(x) = \gamma(xx^{-1}) = \gamma(e) = 1$ and the result follows.
2. Let $x,y \in G$ such that $\gamma_x(y) = 0$ and let $z \in G$,

$$(\gamma_x \cap \gamma_y)(z) = \inf\{\gamma_x(z), \gamma_y(z)\} = \inf\{\gamma(xz^{-1}), \gamma(yz^{-1}\}.$$

On the other hand,

$$\begin{aligned} 0 &= \gamma_x(y) \\ &= \gamma(xy^{-1}) = \gamma((xz^{-1})(zy^{-1})) \geq \inf\{\gamma(xz^{-1}), \gamma(zy^{-1})\} = \inf\{\gamma(xz^{-1}), \gamma(yz^{-1})\}, \end{aligned}$$

since $\gamma(zy^{-1}) = \gamma((zy^{-1})^{-1}) = \gamma(yz^{-1})$. We then deduce that

$$\inf\{\gamma_x(z), \gamma_y(z)\} = \inf\{\gamma(xz^{-1}), \gamma(zy^{-1})\} = 0$$

and the intersection is reduced to the null fuzzy set $\underline{0}$.

3. Let $x, y \in G$ such that $\gamma_x(y) > 0$ and $z \in G$.
   Assume that $z \in Supp(\gamma_x)$ then $\gamma(xz^{-1}) = \gamma_x(z) > 0$, but

   $$\gamma_y(z) = \gamma(xz^{-1}) = \gamma(yx^{-1}xz^{-1}) \geq \inf\{\gamma(yx^{-1}), \gamma(xz^{-1})\} = \inf\{\gamma(xy^{-1}), \gamma(xz^{-1})\} > 0$$

   since both $\gamma(xy^{-1})$ and $\gamma(xz^{-1})$ are strictly positive. Consequently, $\gamma_y(z) > 0$ and $z \in Supp(\gamma_y)$. The same argument can be used to prove that

   $$z \in Supp(\gamma_y) \rightarrow z \in Supp(\gamma_x) > 0.$$

4. For any $y \in G$, we have $1 = \gamma_y(y) \leq (\bigcup_{x\in G} \gamma_x)(y)$, so the result follows. ■

Example 36. G will be the Klein four-group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, γ the fuzzy subgroup given by $\gamma(0,0) = 1,\ \gamma(0,1) = 0.5,\ \gamma(1,0) = 0,\ \gamma(1,1) = 0$ and the relation R is defined by $\forall x,\ y \in G,\ R_\gamma(x,y) = \gamma(x+y)$.

Table 8. Values of γ_a for $a \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

	(0,0)	(0,1)	(1,0)	(1,1)
$\gamma_{(0,0)}$	1	0.5	0	0
$\gamma_{(0,1)}$	0.5	1	0	0
$\gamma_{(1,0)}$	0	0	1	0.5
$\gamma_{(1,1)}$	0	0.5	0	1
$\bigcup_{x\in G} \gamma_x$	1	1	1	1

We remark that

1. $\gamma_{(0,0)} = \gamma$.
2. Every time $\gamma_a(b) = 0$, one has $\gamma_a \cap \gamma_b = \underline{0}$. For example, we have $\gamma_{(0,1)}(1,0) = 0$ and $\gamma_{(0,1)} \cap \gamma_{(1,0)} = \underline{0}$.
3. $Supp(\gamma_x)$ for any $x \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is exactly $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.
4. $\bigcup_{x\in G} \gamma_x = \underline{1}$.

Proposition 37. *Let γ be a normal fuzzy subgroup of the group G. If $z \in Supp(\gamma_x)$ and $t \in Supp(\gamma_y)$, then $zt \in Supp(\gamma_{xy})$.*

Proof. Since γ is normal, we have $\gamma(z(yt^{-1})z^{-1}) = \gamma(yt^{-1}))$ for any $y, z, t \in G$.

On the other hand,

$$\gamma_{xy}(zt) = \gamma(xyt^{-1}z^{-1}) = \gamma(xz^{-1}zyt^{-1}z^{-1}) = \gamma((xz^{-1})(z(yt^{-1})z^{-1}))$$
$$\geq \inf\{\gamma((xz^{-1}), \gamma(z(yt^{-1})z^{-1})\} = \inf\{\gamma((xz^{-1}), \gamma(yt^{-1}))\} = \inf\{\gamma_x(z), \gamma_y(t)\} > 0$$

since $z \in Supp(\gamma_x)$ and $t \in Supp(\gamma_y)$. Finally, $zt \in Supp(\gamma_{xy})$. ■

Definition 38. Let γ be a fuzzy subgroup of a group G. We can define a relation ${}_\gamma R$ defined on G called the left fuzzy relation modulo γ by

$$\forall x, y \in G,\ {}_\gamma R(x, y) = \gamma(y^{-1}x).$$

The left fuzzy coset of x will be denoted by ${}_x\gamma$. Therefore,

$${}_x\gamma : G \longrightarrow [0, 1],\ {}_x\gamma(y) =_\gamma R(x, y) = R_\gamma(y^{-1}, x^{-1}) = \gamma(y^{-1}x).$$

Proposition 39. *Let γ be a normal fuzzy subgroup of a group G. For any $x \in G$, $\gamma_x =_x \gamma$ (i.e., the fuzzy left coset of x is equal to the fuzzy right coset of x).*

Proof. Let $y \in G$, then

$$\gamma_x(y) = \gamma(xy^{-1}) = \gamma(yx^{-1}) = \gamma(x^{-1}(yx^{-1}))x) = \gamma(x^{-1}y) = \gamma(y^{-1}x) =_x \gamma(y).$$

We can also conclude that $G/R_\gamma = G/_\gamma R$. ■

Definition 40. Let us denote by $\mathcal{F}(G)$ and $\mathcal{F}(G')$ the sets of fuzzy sets on G and G' respectively. If $\mu \in \mathcal{F}(G)$ and $\gamma \in \mathcal{F}(G')$, we define the mapping

$$\mu \otimes \gamma : G \times G' \ \rightarrow\ [0, 1],\ (x, y)\ \mapsto\ \mu(x)\gamma(y).$$

If $A \subset \mathcal{F}(G),\ B \subset \mathcal{F}(G')$, by extension we define $A \otimes B = \{\mu \otimes \gamma,\ \mu \in A,\ \gamma \in B\}$.

Definition 41. Let γ be a fuzzy subgroup of a group G. We define a binary operation on $G/R_\gamma \times G/R_\gamma$ by

$$\begin{array}{rccccc} \otimes_\gamma : & G/R_\gamma \times G/R_\gamma & \rightarrow & G/R_\gamma \otimes G/R_\gamma & & \\ & (\gamma_x, \gamma_y) & \mapsto & \gamma_x \otimes_\gamma \gamma_y : & G \times G \ \rightarrow & [0,1] \\ & & & & (z,t) \ \mapsto & \gamma_x(z).\gamma_y(t). \end{array}$$

The mapping $\gamma_x \otimes_\gamma \gamma_y$ will denotes by $\gamma_{x \otimes_\gamma y}$.

Proposition 42. *If $x' \in Supp(\gamma_x)$ and $y' \in Supp(\gamma_y)$ then*

$$Supp(\gamma_{x \otimes_\gamma y}) = Supp(\gamma_{x' \otimes_\gamma y'}).$$

Proof. Let $(z, t) \in Supp(\gamma_{x \otimes_\gamma y})$ then $\gamma_x(z)\gamma_y(t) > 0$, so $\gamma_x(z) > 0$ and $\gamma_y(t) > 0$. On the other hand,

$$\gamma_{x'}(z) = \gamma(x'z^{-1}) = \gamma(x'x^{-1}xz^{-1}) \geq \min\{\gamma(x'x^{-1}), \gamma(xz^{-1}) > 0$$

since $\gamma(x'x^{-1}) = \gamma(xx'^{-1}) > 0$ ($x' \in Supp(\gamma_x)$) and $\gamma(xz^{-1}) > 0$. By the similar arguments, we prove that $\gamma_{y'}(t) > 0$. As $(\gamma_{x' \otimes_\gamma y'})(z, t) \geq \min\{\gamma_{x'}(z), \gamma_{y'}(t)\} > 0$, we can conclude that $(z, t) \in Supp(\gamma_{x' \otimes_\gamma y'})$ and then

$$Supp(\gamma_{x \otimes_\gamma y}) \subset Supp(\gamma_{x' \otimes_\gamma y'}).$$

The other inclusion is then trivial. ■

Example 43. Coming back to Example 36, let $x = (1, 1)$, $x' = (1, 1)$, $y = (0, 1)$ and $y' = (0, 0)$. It is easy to see from the tables that $x' \in Supp(\gamma_x)$, $y' \in Supp(\gamma_y)$. On the other hand, $Supp(\gamma_{x \otimes_\gamma y}) = \{((0, 1), (0, 1))\} = Supp(\gamma_{x' \otimes_\gamma y'})$.

4. CONCLUSION

We have revisited some concepts that we investigated in our previous works. We refined some of the results that we thought were a bit confusing or a little unclear. As a second goal, we introduced the notion of fuzzy relation modulo a fuzzy subgroup and then fuzzy right coset and fuzzy left coset modulo a subgroup. We proved that when the fuzzy subgroup is normal, the fuzzy right and fuzzy left cosets are equal.

Our future interest is to investigate the fuzzy quotient set obtained from a ring and a fuzzy relation modulo a fuzzy ideal. More exactly, we want to answer the question whether the fuzzy quotient set can be endowed with a fuzzy ring structure.

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