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REVIEW

A NOVEL EFFICIENT APPROACH TO THE SOLUTION OF NONLINEAR BOUNDARY VALUE PROBLEMS

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Abstract. This paper provides an overview of a novel efficient approach to study both qualitative aspects and numerical solution of boundary value problems (BVPs) for high order nonlinear differential equations developed by the author Dang Quang A and his collaborators recently. This approach is also extended from BVPs with two-point boundary conditions to integral BVPs, and from ordinary differential equations (ODEs) to integral differential equations, functional differential equations and partial differential equations. Our published works to date demonstrate the efficiency of the approach in comparison with that of some existing methods. The approach is general and it can be applied to other nonlinear BVPs.

Keywords. High order boundary value problem, integro-differential equation, functional differential equation, biharmonic equation, existence and uniqueness of solution, numerical method.

1. INTRODUCTION

Numerous problems in the fields of mechanics, physics, chemical engineering, biology, environment, etc. are reduced to boundary value problems for second or high order linear or nonlinear ordinary differential equations (ODEs) (see, e.g., [1, 2]), integro-differential equations (IDE) (see, e.g., [3, 4]) and functional differential equations (FDE) (see, e.g., [5, 6]). One can find their exact solutions in a very small number of special cases. In general, one needs to seek their approximations by approximate methods, mainly numerical methods.

It can be stated that among high order ODEs the fourth order and the third order nonlinear differential equations are more attractive due to their wide range of applications. These include the beam theory, modeling of the deflection of a curved beam having a constant or varying cross-section, three-layer beam, electromagnetic wave incident on a system of charges sets, the regulation of a steam turbine and so on.

Differential equations of higher orders also attract attention from researchers because they arise from many physical problems. For examples, fifth order ODE arises in induction

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motor models [7], sixth order ODE models problems in astrophysics [8], seventh order ODE arises in sculpturing electrical process of motors with two rotating mechanism circuit [9], eighth order ODE is derived from governing bending and axial vibrations [10] and so on.

A general statement of a boundary value problem (BVP) for nth order nonlinear ODEs is as follows:
Find a function $u(x)$ satisfying the equation

$$L_n u \equiv u^{(n)}(x) = f(x, u(x), u'(x), ..., u^{(n-1)}(x)),\ x \in (0,1), \tag{1}$$

and the boundary conditions

$$B_i[u] = c_i,\ (i = 1, ..., n), \tag{2}$$

where $B_i, c_i\ \ (i = 1, ..., n)$ are linear boundary operators and real numbers, respectively.

Often met cases of the above general BVP are:
- BVP for third order ODE [11]

$$\begin{aligned} &u'''(t) = f(t, u(t), u'(t), u''(t)),\ 0 < t < 1,\\ &B_1[u] = \alpha_1 u(0) + \beta_1 u'(0) + \gamma_1 u''(0) = 0,\\ &B_2[u] = \alpha_2 u(0) + \beta_2 u'(0) + \gamma_2 u''(0) = 0,\\ &B_3[u] = \alpha_3 u(1) + \beta_3 u'(1) + \gamma_3 u''(1) = 0, \end{aligned} \tag{3}$$

such that

$$\operatorname{rank}\begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 & 0 & 0 & 0\\ \alpha_2 & \beta_2 & \gamma_2 & 0 & 0 & 0\\ 0 & 0 & 0 & \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix} = 3.$$

The problem (3) includes as particular cases the problems considered in [12, 13, 14, 15, 16].

- BVP for fourth order ODE [17]

$$\begin{aligned} &u^{(4)}(x) = f(x, u(x), u'(x), u''(x), u'''(x)),\ 0 < x < 1,\\ &B_1(\bar{u}) = 0,\ B_2(\bar{u}) = 0,\\ &B_3(\bar{\bar{u}}) = 0,\ B_4(\bar{\bar{u}}) = 0, \end{aligned} \tag{4}$$

where B_1, B_2, B_3, B_4 are linear combinations of the components of arguments

$$\begin{aligned} \bar{u} &= \big(u(0), u(1), u'(0), u'(1)\big),\\ \bar{\bar{u}} &= \big(u''(0), u''(1), u'''(0), u'''(1)\big). \end{aligned}$$

The problem (4) includes particular cases previously studied in [18, 19, 20, 21, 22, 23].

A number of results are devoted to the existence, uniqueness and positivity of solutions of the problems (3) and (4) with different specific boundary conditions. The methods for investigating qualitative aspects of the problems are diverse, including the method of lower and upper solutions and monotone technique (see, e.g., [21, 22, 23, 24, 25, 26, 27, 28]), Schauder and Banach fixed point theorems [1], the fixed point index theory in cones [29], the Leray-Schauder continuation principle [14], Fourier analysis [30], the reproducing kernel theorem [31], and so on. It should be emphasized that in almost above works there is an essential assumption that the function $f(x, y, u, v)$ in (3) or $f(x, y, u, v, w)$ in (4) satisfies a Nagumo-type condition on the last two variables, or linear growth at infinity, or some complicated conditions including monotone increase in each of variables. *These drawbacks were overcome by our novel approach which will be presented later in the next section.*

For the applied purpose, the finding the solution of the nonlinear BVPs plays very important role. There are many methods for solving nonlinear BVPs for ODEs including analytical and numerical methods. Among the analytical methods there are variational iteration method [32, 33, 34, 35], Picard iterative method [1], Adomian decomposition methods (ADM) [36, 37, 38], combination of ADM with Green's function [39], Differential Transform Methods [40], and so on. In general, the analytical methods are efficiently applicable when the right hand side function is simple so that the integrals containing it are easy to be calculated. Therefore, in practice many numerical methods are developed for solving nonlinear BVPs for high order ODEs. The main numerical methods used are the collocation methods (see. e.g., [41, 42, 43, 44, 45, 46, 47]), finite difference methods [48, 49, 50], spectral-Galerkin method [51]. These methods are applied directly to the nonlinear BVPs and result to nonlinear system of algebraic equations which need to be solved by approximate methods one of them is the popular Newton method. The nonlinear system of algebraic equations in essence is the result of the discretization of the original BVPs and in the above mentioned works some error estimates of the methods were proved to be $O(h^2)$ or $O(h^4)$, where h is the grid size. However, the estimate of the total error of the actual numerical solution was not obtained because the error when using the Newton method was omitted. Besides, when constructing numerical methods for solving BVPs the authors always assumed that the problems have unique sufficiently smooth solutions or neglected this issue. *In the next section we will review our results of the novel approach to nonlinear BVPs which considers the existence and uniqueness of solutions and the method for constructing approximate solutions along with the estimate of total error of the actually obtained numerical solutions.*

Above we are concerned with two-point nonlinear BVPs. They are local ones. Recently, many authors considered nonlocal problems, where the boundary conditions contain the integral of the function to be sought or its derivatives. These problems are named integral BVPs or BVPs with integral boundary conditions. This type of problems appears in applied fields such as flexibility mechanics, chemical engineering, thermodynamics,... There have been many publications on boundary value problems with integral boundary conditions such as [52, 53, 54, 55, 56, 57, 58]. Using different versions of fixed point theory the authors established the existence, nonexistence and multiplicity of solutions of the problems. A limitation of these results is that sufficient conditions for the existence of solutions were established with together some examples of satisfaction of these conditions but no solutions were shown. The numerical solution of integral BVPs has not attracted yet attention from researchers. To our best knowledge, only in [58] Pandey proposed the finite difference method of the accuracy $O(h^2)$ for solving the third order BVP with integral boundary conditions. *In the next section we shall briefly touch the investigation of some integral BVPs in both qualitative and quantitative aspects.*

Alongside with ODEs integro-differential equations (IDEs) are mathematical models of many phenomena in physics, biology, fluid mechanics, chemistry, epidemiology (see, e.g., [4, 59, 60]). To our best knowledge, there is only few works concerning the theoretical study of qualitative aspects of IDEs and not litle works of analytical approximation and numerical methods have been developed for these equations (see, e.g., [61, 62, 63, 64, 65, 66]) and bibliography therein. To investigate the existence and uniqueness of solution and numerical method for finding the solution of a fourth order nonlinear IDE [67] we also successfully applied our method developed in [18, 20, 68].

Except for the ODEs and IDEs mentioned above, in this overview we also concern with functional differential equations (FDEs). They are equations of the form

$$u^{(k)}(x) = f(x, u(x), u'(x), ..., u^{(k-1)}(x), u(\varphi(x)), \tag{5}$$

where f and φ is are given functions. FDEs appear in many application fields such as electrodynamics, nonlinear dynamics systems, quantum mechanics, astrophysics, biology,... (see [5]). When the function is a proportional delay function, that is, when $\varphi(x) = \alpha x$, $0 < \alpha < 1$ the equation is called the pantograph equation. The name comes from the paper by Ockendon and Tayler [69] when considering the motion of the jointed framework conveying a current to an electric locomotive. So far, there have been many publications on boundary value problems for functional differential equations. Numerical methods for solving problems of this type are quite diverse such as projection and using polynomial splines method [70], shooting method [71] , Picard and Mann iterative schemes based on Green function [72], neural networks [73, 74]. Recently, for solving BVPs for FDEs Bica et al. [75, 76] constructed successive approximations for the equivalent integral equation with the use of cubic spline interpolation at each iterative step. They established maximal order of convergence of the method on dependence of the order of the FDEs. *For solving BVPs for FDEs in [77, 78] we successfully applied the method which was used for solving ODEs and IDEs.*

Besides ODEs, IDEs and FDEs related to one-dimensional equations our approach was used to two-dimensional elliptic equations, namely, to nonlinear biharmonic and triharmonic equations which describe the static deflection of an elastic bending plate [67, 79, 80].

In this paper we review our results of the existence, uniqueness and numerical methods for solving ODEs, IDEs, FDEs and PDEs obtained by using a unified novel approach.

The structure of the paper is as follows. After the introduction, Section 2 briefly describes the general methodology of our approach to investigate the existence of solution and iterative method on continuous level for ODEs. In Section 3 we present the construction of high order numerical methods for solving nonlinear ODEs based on the discretization of the continuous iterative methods using trapezoidal formula with corrections. In Section 4 we review some results of applications of the approach to IDEs. The applications of the approach to some FDEs and PDEs are presented in Sections 5 and 6, respectively. Finally, in Section 7 we conclude the paper.

2. NOVEL METHOD FOR INVESTIGATING NONLINEAR BVPs FOR ODEs: EXISTENCE OF SOLUTION AND ITERATIVE METHOD ON CONTINUOUS LEVEL

2.1. General methodology

We begin this section by stating the general idea of our approach to the BVP (1)-(2) with homogeneous boundary conditions. For the purpose of easy tracking we rewrite the problem in the form

$$
\begin{aligned}
u^{(n)}(x) &= f(x, u(x), u'(x), ..., u^{(n-1)}(x)),\ x \in (0,1),\\
B_i[u] &= 0,\ (i = 1, ..., n).
\end{aligned}
\tag{6}
$$

We shall associate this problem with an operator equation. For functions $\varphi(x) \in C[0,1]$ consider the nonlinear operator A defined by

$$
(A\varphi)(x) = f(x, u(x), u'(x), ..., u^{(n-1)}(x)),
\tag{7}
$$

where $u(x)$ is the solution of the problem

$$
\begin{aligned}
u^{(n)}(x) &= \varphi(x),\ x \in (0,1),\\
B_i[u] &= 0\ (i = 1, ..., n),
\end{aligned}
\tag{8}
$$

provided that it is uniquely solvable. It is easy to verify the following proposition.

Proposition 2.1. *If the function $\varphi(x)$ is a fixed point of the operator A, i.e., $\varphi(x)$ is a solution of the operator equation*

$$
A\varphi = \varphi,
\tag{9}
$$

then the function $u(x)$ determined from the boundary value problem (8) solves the problem (6). Conversely, if $u(x)$ is a solution of the boundary value problem (6) then the function

$$
\varphi(x) = f(x, u(x), u'(x), ..., u^{(n-1)}(x))
$$

is a fixed point of the operator A defined above by (7), (8).

Thus, the solution of the original problem (6) is reduced to the solution of the operator equation (9).

This proposition plays a key role in our approach to nonlinear BVPs. It was proved and used for the cases $n = 4$ in [18, 20, 68, 81] and for $n = 3$ in [11], for $n = 6$ in [82].

Let $G_0(x,s)$ be the Green function [83] associated with the problem (8). Then the solution of this problem is represented in the form

$$
u(x) = \int_0^1 G_0(x,s)\varphi(s)ds.
\tag{10}
$$

By differentiation of both sides of the above formula we obtain

$$
u^{(k)}(x) = \int_0^1 G_k(x,s)\varphi(s)ds,\ (k = 1, ..., n-1)
\tag{11}
$$

where $G_k(x,s) = \frac{\partial^k G_0(x,s)}{\partial x^k}$. Notice that $G_k(x,s)$ $(k = 0, 1, ..., n-2)$ are functions continuous in the square $Q = [0,1]^2$ and $G_{n-1}(x,s)$ is discontinuous in the square Q except for the line $x = s$.

Further, let

$$
\max_{0 \le x \le 1} \int_0^1 |G_k(x,s)|ds = M_k,\ k = 0, ..., n-1.
\tag{12}
$$

Next, for each positive real number $M > 0$ introduce the domain

$$\mathcal{D}_M = \{(x, y, y_1, ..., y_{n-1})|\ 0 \le x \le 1,\ |y| \le M_0 M, |y_k| \le M_k M,\ k = 1, ..., n-1\},$$

and as usual, by $B[O, M]$ we denote the closed ball of radius M centered at 0 in the space $C[0, 1]$ of continuous in $[0, 1]$ functions, namely,

$$B[O, M] = \{\varphi \in C[0, 1]|\ \|\varphi\| \le M\},$$

where

$$\|\varphi\| = \max_{0 \le x \le 1} |\varphi(x)|.$$

The existence of solutions is guarantied by the following theorem.

Theorem 2.2. *[Existence of solutions] Suppose that there exists a number $M > 0$ such that the function $f(x, y_1, ..., y_{n-1})$ is continuous and bounded by M in the domain $\mathcal{D}_M$, i.e.,*

$$|f(x, y, y_1, ..., y_{n-1})| \le M \tag{13}$$

for any $(x, y, y_1, ..., y_{n-1}) \in \mathcal{D}_M$.

Then, the problem (6) has a solution $u(x)$ satisfying

$$|u(x)| \le M_0 M,\ |u^{(k)}(x)| \le M_k M,\ \text{for any } 0 \le x \le 1. \tag{14}$$

The tool for proving the theorem is the Schauder Fixed Point Theorem [84] applied to the operator A defined by (7) and (8) above.

Theorem 2.3. *[Existence and uniqueness of solution] Assume that there exist numbers $M > 0, L_k \ge 0$ $(k = 0, 1, ..., n-1)$ such that*

$$|f(x, y, y_1, ..., y_{n-1})| \le M,\ \forall (x, y, y_1, ..., y_{n-1}) \in \mathcal{D}_M,$$

$$|f(x, y', y'_1, ..., y'_{n-1}) - f(x, y'', y''_1, ..., y''_{n-1})| \le L_0|y' - y''| + \sum_{k=1}^{n-1} L_k |y'_k - y''_k| \tag{15}$$

for any $(x, y', y'_1, ..., y'_{n-1}), (x, y'', y''_1, ..., y''_{n-1}) \in \mathcal{D}_M$ and

$$q := \sum_{k=0}^{n-1} L_k M_k < 1. \tag{16}$$

Then, the problem (6) has a unique solution $u(x)$ such that $|u(x)| \le M_0 M$, $|u^{(k)}(x)| \le M_k M$, $(k = 1, ..., n-1)$ for any $0 \le x \le 1$.

The way to prove the theorem is to show that the operator A is a contraction mapping from $B[0, M]$ into itself and apply the Banach fixed theorem.

The the successive approximation of the fixed point of the operator A associated with the problem (6) is reduced to the following iterative method for solving the problem:

1. Given a starting approximation $\varphi_0 \in B[0, M]$, say

$$\varphi_0(x) = 0. \tag{17}$$

2. Knowing $\varphi_m(x)$ $(m = 0, 1, ...)$ compute

$$\begin{aligned} u_m(x) &= \int_0^1 G_0(x,s)\varphi_m(s)\, ds, \\ y_k^{(m)}(t) &= \int_0^1 G_k(x,s)\varphi_m(s)\, ds,\ (k = 1, ..., n-1). \end{aligned} \tag{18}$$

3. Update the new approximation

$$\varphi_{m+1}(x) = f(t, u_m(t), y_1^{(m)}(x), ..., y_{n-1}^{(m)}(x)). \tag{19}$$

Theorem 2.4. *[Convergence] Under the assumptions of Theorem 2.3 the above iterative method converges and there hold the estimates*

$$\|u_m - u\| \le M_0 p_m, \quad \|u_m^{(k)} - u^{(k)}\| \le M_k p_m,\ k = 1, ..., n-1;\ m = 1, 2, ... \tag{20}$$

where u is the exact solution of the problem (6), M_k $(k = 0, 1, ..., n-1)$ are given by (12) and

$$p_m = \frac{q^m}{1-q}\|\varphi_1 - \varphi_0\|. \tag{21}$$

Now consider some applications of the above theory for nth order nonlinear BVPs. It is through specific problems for nonlinear ODEs of orders 3, 4, 5, 6 that the general theory presented above is formed and developed.

2.2. Applications to third order BVPs

In [11] the above theory of the existence of solutions and the iterative method was obtained for third order nonlinear BVPs. There, we also obtained results of the existence of positive solutions and monotone solutions, too. Many examples showed that the results obtained by using the above theory are better ones in [14, 12, 13, 15, 16].

2.3. Applications to fourth order BVPs

In [81] we considered the problem

$$\begin{aligned} u^{(4)}(x) &= f(x, u(x), u'(x), u''(x), u'''(x)),\ a < x < b, \\ u(a) &= u(b) = 0,\ u'(a) = u'(b) = 0, \end{aligned} \tag{22}$$

which describes the deformations of an elastic beam with both fixed end-points. For this problem the Green function has the form

$$G(x,t) = \frac{1}{6(b-a)^2} \begin{cases} (x-a)^2(b-t)^2\Big[(t-x) + \dfrac{2(b-x)(t-a)}{b-a}\Big], \\ \quad a \le x \le t \le b, \\ (t-a)^2(b-x)^2\Big[(x-t) + \dfrac{2(x-a)(b-t)}{b-a}\Big], \\ \quad a \le t \le x \le b, \end{cases}$$

and

$$M_k = \max_{0 \le x \le 1} \int_a^b \Big| \frac{\partial^k G(x,t)}{\partial x^k} \Big| dt = C_k (b-a)^{4-k}, \ k = 0,1,2,3,$$

where $C_0 = 1/384, \quad C_1 = 1/72\sqrt{3}, \quad C_2 = 1/12, \quad C_3 = 1/2.$
As shown in [81] our theory described above gives results somewhat better than in [85] on some examples.
In [17] when considering the problem (4) the operator A is defined as

$$(A\varphi)(x) = f(x, u(x), u'(x), u''(x), u'''(x)), \tag{23}$$

where $u(x)$ is the solution of the problem

$$\begin{aligned} &u^{(4)}(x) = \varphi(x), \ 0 < x < 1, \\ &B_1(\bar{u}) = 0, \ B_2(\bar{u}) = 0, \\ &B_3(\bar{\bar{u}}) = 0, \ B_4(\bar{\bar{u}}) = 0, \end{aligned} \tag{24}$$

with

$$\begin{aligned} \bar{u} &= \big(u(0), u(1), u'(0), u'(1)\big), \\ \bar{\bar{u}} &= \big(u''(0), u''(1), u'''(0), u'''(1)\big). \end{aligned}$$

In this case we set $v(x) = u''(x)$. Then it is decomposed into two second order problems

$$\begin{cases} v''(x) = \varphi(x), \quad 0 < x < 1, \\ B_3(\bar{v}) = 0, \ B_4(\bar{v}) = 0, \end{cases} \tag{25}$$

$$\begin{cases} u''(x) = v(x), \quad 0 < x < 1, \\ B_1(\bar{u}) = 0, \ B_2(\bar{u}) = 0, \end{cases} \tag{26}$$

where $\bar{v} = \big(v(0), v(1), v'(0), v'(1)\big)$. Via these second order BVPs we established the existence, uniqueness, positivity and monotoncity of solutions. The obtained results were shown on many examples are better than those in [21, 22, 23] and some other works.
Especially, when the boundary conditions for the fourth order equation which models a cantilever beam in equilibrium state has the form

$$u(0) = u'(0) = u''(1) = u'''(1) = 0, \tag{27}$$

the problems (25), (26) become initial value problem and final value problem, respectively. Therefore, at each step of the iterative method for the problem it is needed to solve an initial value problem and a final value problem for second order equation. The problem was investigated in [18].

The simplest case of (4) is the problem

$$\begin{aligned} u^{(4)}(x) &= f(x, u(x), u''(x)), \ 0 < x < 1, \\ u(0) &= u(1) = u''(0) = u''(1) = 0. \end{aligned} \tag{28}$$

where $f : [0,1] \times \mathbb{R}^2$ is continuous. This problem models the bending equilibrium of a beam on an elastic foundation, whose two ends are simply supported. It was in [68] the idea of reducing the problem (28) to a sequence of second-order problems was formed by converting the problem into an operator equation for the right-hand side (or nonlinear term). Somewhat later the BVP for fully fourth order ODE with the Lidstone boundary conditions as in (28) was considered in [19].

All considered fourth order problems so far are local, when the right-hand side or nonlinear term has the form $f(x, u(x), u'(x), u''(x), u'''(x))$ and the approach of reducing BVPs to operator equation works well. Furthermore, this approach works for nonlocal problems. Namely, in [86] we applied it successfully to the problem which models the bending equilibrium of an extensible beam

$$\begin{aligned} &u^{(4)}(x) - M\Big(\int_0^L |u'(s)|^2 ds\Big)u''(x) = f(x, u(x), u'(x), u''(x), u'''(x)),\ 0 < x < L, \\ &\quad u(0) = u(L) = 0,\ u''(0) = u''(L) = 0, \end{aligned} \tag{29}$$

by setting

$$\varphi(x) = M(\|u'\|_2^2)u''(x) + f(x, u(x), u'(x), u''(x), u'''(x)),$$

and reduced the BVP to operator equation for φ. Above $\|.\|_2$ denotes the norm of $L^2[0, L]$.

Extending the technique in [86] and using the Brouwer degree theory [87], in [88] we established the existence of solutions of a nonlocal fourth order equation of Kirchhoff type with nonlinear boundary conditions

$$\begin{aligned} &u^{(4)}(x) - M\Big(\int_0^L |u'(s)|^2 ds\Big)u''(x) = f(x, u(x), u'(x)),\ 0 < x < L, \\ &u''(0) = u''(L) = 0, \\ &u'''(0) = -g(u(0)),\ u'''(L) = g(u(L)), \end{aligned}$$

and proposed an iterative method for finding the solutions. Although the iterative method for the problem is not justified but in practice it can be used for finding a solution of it.

The approach to nonlinear BVPs presented in the beginning of the section is not only applied to BVPs but it can also be applied to systems of ODEs. In [89] we used this approach to investigate the solvability and iterative solution of coupled beams equations with fully nonlinear terms.

It must be said that the idea of reducing the high order BVPs to operator equations for the right-hand sides for investigating the existence of solutions as well as building iterative methods to solve them appeared in our earlier paper [90] when studying the Neumann boundary value problem for linear biharmonic equation.

2.4. Applications to fifth and sixth order BVPs

Except for applying the unified methodology to third and fourth order nonlinear BVPs we also used it for fifth and sixth order BVPs. Namely, recently in [91] we considered the fifth order problem

$$\begin{aligned} &u^{(5)}(x) = f(x, u(x)),\ x \in (0, 1), \\ &u(0) = \alpha_0, u'(0) = \alpha_1,\ u''(0) = \alpha_2, \\ &u(1) = \beta_0,\ u'(1) = \beta_2. \end{aligned} \tag{30}$$

The solution of the problem (30) was studied by different methods such variational iteration method [92], finite difference method [93], B-spline collocation method [94], boundary value method [95] and so on. In the work [91] we established the existence and uniqueness of solution of the above problem and proposed the following iterative method for finding its solution

1. Given
$$\varphi_0(x) = f(x, 0). \tag{31}$$

2. Knowing $\varphi_k(x)$ $(k = 0, 1, ...)$ compute
$$u_k(x) = P(x) + \int_0^1 G(x, s)\varphi_k(s)ds. \tag{32}$$

3. Update
$$\varphi_{k+1}(x) = f(x, u_k(x)). \tag{33}$$

Based on this iterative method and the general methodology of constructing high order discrete methods, which will be presented in the next section, we constructed a numerical method of $O(h^8)$ accuracy for the problem (30). This result is better than that in the mentioned above and some other papers.

For sixth order nonlinear BVP of the special form, namely, for the problem

$$\begin{aligned} u^{(6)}(x) &= f(x, u(x), u''(x), u^{(4)}(x)),\ 0 < x < 1, \\ u(0) &= u''(0) = u^{(4)}(0) = 0, \\ u(1) &= u''(1) = u^{(4)}(1) = 0 \end{aligned} \tag{34}$$

by reduction of it to an operator equation for the right-hand side we also established results on existence, uniqueness of solution and convergence of the iterative method which leads the solution of the problem to the solution of three second order problems at each iteration. To each second order BVP applying sixth order difference scheme we obtained the accuracy $O(h^6)$ for the sixth order nonlinear BVP. This result was published in [82].

3. CONSTRUCTION OF HIGH ORDER NUMERICAL METHODS FOR ODEs

3.1. General methodology

In the previous section have presented a novel unified approach to investigate nonlinear BVPs for ODEs. It is based on the reduction of BVPs to operator equations for right-hand sides or nonlinear terms while others authors usually reduced them to operator equation for the functions to be searched. Except for the advantages in investigating qualitative aspects of the BVPs this approach gives a unified method for constructing high order numerical methods. The method is based on the design of formulas of high order accuracy for computing the integrals $\int_0^1 G_0(x, s)\varphi_m(s)\ ds$ and $\int_0^1 G_k(x, s)\varphi_m(s)\ ds$ in the step 2 of the iterative method on continuous level (18). For the purpose we use the formula Euler-Maclaurin (see [96]) for computing the integral of functions having jumps on the uniform grid $\overline{\omega}_h = \{x_i = (i-1)h,\ i = 1, ..., N+1;\ h = 1/N\}$

$$\begin{aligned}\int_0^1 \Phi(s)ds =& T_\Phi(h) - \sum_{l=1}^{p-1} \frac{B_{2l}}{(2l!)} \Big\{ [\Phi^{(2l-1)}(1) - \Phi^{(2l-1)}(0)] \\ & - [\Phi^{(2l-1)}(t^+) - \Phi^{(2l-1)}(t^-)] \Big\} + O(h^{2p}),\end{aligned}$$

where B_{2l} are Bernoulli numbers, $\psi(t^+)$ and $\psi(t^-)$ are the one-sided limits of the function $\psi(s)$ at t,

$$T_\Phi(h) = \frac{h}{2}(\Phi_1 + \Phi_{N+1}) + \sum_{i=2}^{N} h\Phi_i.$$

Using the above formula we propose discrete iterative method for solving the problem (6). Denote by $\Phi_m(x), U_m(x), Y_k^{(m)}(x)$ the grid functions, which are defined on the grid $\bar{\omega}_h$ and approximate the functions $\varphi_m(x), u_m(x), y_k^{(m)}(x)$ on this grid, respectively. Consider the following discrete iterative methods called p-Iterative Method for numerically realizing the iterative method on continuous level (17)-(19):

1. Given
$$\Phi_0(x_i) = f(x_i, 0, ..., , 0),\ i = 1, ..., N+1.$$

2. Knowing $\Phi_m(x_i),\ m = 0, 1, ...;\ i = 1, ..., N+1$, compute approximately the definite integrals (2) by the trapezoidal formulas with corrections
$$\begin{aligned} U_m(x_i) &= L_{2p}(G_0, x_i)\Phi_m, \\ Y_k^{(m)}(x_i) &= L_{2p}(G_k, x_i)\Phi_m,\ i = 1, ..., N+1, \end{aligned} \tag{35}$$
where $L_{2p}(G_0, x_i)\Phi_m$, $L_{2p}(G_k, x_i)\Phi_m$ are the approximations of the intergrals
$$\begin{aligned} &\int_0^1 G_j(x_i, s)\varphi(s)ds = L_{2p}(G_j, x_i)\varphi + O(h^{2p}), \\ &j = 0, 1, ..., n-1;\ i = 1, ..., N+1;\ p = 2, 3, 4. \end{aligned}$$

3. Update
$$\Phi_{k+1}(x_i) = f(x_i, U_k(x_i), Y_k^{(1)}(x_i), ..., Y_k^{(n-1)}(x_i)),\ i = 1, ..., N+1. \tag{36}$$

The convergence of the discrete iterative method is given by the following theorem.

Theorem 3.1. *Under the assumptions of Theorem 2.4 for the approximate solution of the problem (6) obtained by the discrete p-Iterative Method ($p = 2, 3, 4$) on the uniform grid with gridsize h we have the estimates*

$$\|U_m - u\| \le M_0 p_m d + O(h^{2p}),\ \|Y_k^{(m)} - u^{(k)}\| \le M_k p_m d + O(h^{2p}),\ p = 2, 3, 4,$$

where M_0, M_k are defined by (12) and p_m are defined by (21).

3.2. Application to third order nonlinear BVPs

Using the above general methodology, very recently in [97] we constructed numerical methods of $O(h^4), O(h^6)$ and $O(h^8)$ accuracy for the typical third order BVP

$$\begin{aligned} u^{(3)}(x) &= f(x, u(x), u'(x), u''(x)),\ 0 < x < 1, \\ u(0) &= c_1, u'(0) = c_2, u'(1) = c_3. \end{aligned} \tag{37}$$

To the best of our knowledge it is the first time for the third order nonlinear BVP a method having the errors $O(h^8)$ for both the solution and its derivatives has been constructed. In [98] by two-step hybrid block method with fourth derivatives the authors constructed a method of seventh order convergence (37) but the estimate was obtained only for the solution but not for its derivatives.

Remark that in [99] using the trapezoidal formula (without corrections) with linear interpolation we constructed a method of $O(h^3)$ accuracy for the problem (37).

As an application we used the sixth order method to solve numerically the third order obstacle problem

$$\begin{aligned} u''' &= f(x, u),\ \ 0 \le x \le 1, \\ u(0) &= u'(0) = u'(1) = 0, \end{aligned} \tag{38}$$

where

$$f(x,u) = \begin{cases} 0, & 0 \le x \le \frac{1}{4} \text{ and } \frac{3}{4} \le x \le 1, \\ u - 1, & \frac{1}{4} \le x \le \frac{3}{4}. \end{cases} \tag{39}$$

Until our work some authors such Khan and Sultana, Gao and Chi, Pandey and so on solved this problem by different methods with accuracy not greater order 4.

3.3. Application to fourth order nonlinear BVPs

Using also the general methodology presented in Subsection 3.1 we constructed high order numerical methods for solving fourth order nonlinear BVP

$$\begin{aligned} u^{(4)}(t) &= f(t, u(t), u'(t), u''(t), u'''(t)),\ 0 < t < 1, \\ u(0) &= a,\ u(1) = b,\ u'(0) = c,\ u'(1) = d. \end{aligned} \tag{40}$$

It is possible to say that the 4-Iterative method, i.e., the method with $O(h^8)$ accuracy is better all existing numerical methods in the sense of accuracy of the solution and its derivatives and its implementation. This result was published in the recent work [100].

Remark that for the BVP of the form

$$\begin{aligned} &u^{(4)}(x) = f(x, u(x), u''(x)),\ 0 < x < 1, \\ &u(0) = 0,\ u(1) = 0,\ u''(0) = 0,\ u''(1) = 0, \end{aligned}$$

by the conversion of it to an operator equation for the right-hand side and setting $v(x) = u''(x)$ the problem was reduced to the solution of a sequence of second order BVPs. In [101] we designed $O(h^6)$ difference schemes for the latter second BVPs. So, in result we constructed sixth order method for the above problem.

3.4. Application to second order nonlinear BVPs

The second order nonlinear BVP of the general type

$$\begin{aligned} &u''(t) = f(t, u(t), u'(t)),\ 0 < t < 1, \\ &au(0) - bu'(0) = \alpha,\ cu(1) + du'(1) = \beta, \end{aligned}$$

where the function $f(t, u, y)$ is assumed to be sufficiently smooth, α, β are real constants, $a, b, c, d \geq 0$, $\rho = ad+bc+ac > 0$, is very often met in various fields of science and engineering. Therefore, there are many analytical and numerical methods for solving it. But maximal order of accuracy of the existing numerical methods is 7. Very recently, in [102] by the general methodology described in Subsection 3.1 we have constructed a numerical method which has the accuracy $O(h^8)$. The method is applied to solve the famous Bratu and Bratu-type problems and the obstacle problems, showing the efficiency of the proposed method in comparison with some existing methods.

4. INTEGRAL BVPs AND INTEGRO-DIFFERENTIAL EQUATIONS

4.1. Integral BVPs for ODEs

The general methodology for investigating nonlinear BVPs for ODEs presented in Subsection 2.1 can be extended to integral BVPs. In [103, 104, 105] we developed the method to reduce the integral BVPs to operator equation for the right-hand side of the equations and the integral in boundary conditions. For illustration of the method to integral BVPs consider the problem (see [103])

$$\begin{aligned} &u''''(t) = f(t, u(t), u'(t), u''(t), u'''(t)),\ 0 < t < 1, \\ &u'(0) = u''(0) = u'(1) = 0,\ u(0) = \int_0^1 g(s)u(s)ds, \end{aligned} \tag{41}$$

where $f : [0, 1] \times \mathbb{R}^4 \to \mathbb{R}^+,\ g : [0, 1] \to \mathbb{R}^+$ are continuous functions.

To investigate the problem (41) we associate it with an operator equation as follows.

First, we denote the space of pairs $w = (\varphi, \mu)^T$, where $\varphi \in C[0, 1], \mu \in \mathbb{R}$, by $\mathcal{B}$, i.e., set $\mathcal{B} = C[0, 1] \times \mathbb{R}$, and equip it with the norm

$$\|w\|_{\mathcal{B}} = \max(\|\varphi\|, r|\mu|),$$

where r is a number $r \geq 1$ to be determined later for each specific problem and $\|\varphi\| = \max_{0 \leq t \leq 1} |\varphi(t)|$.

Further, define the operator A acting on elements $w \in \mathcal{B}$ by the formula

$$Aw = \begin{pmatrix} f(t, u(t), u'(t), u''(t), u'''(t)) \\ \int_0^1 g(s)u(s)ds \end{pmatrix}, \tag{42}$$

where $u(t)$ is the solution of the problem

$$\begin{aligned} &u''''(t) = \varphi(t),\ 0 < t < 1, \\ &u'(0) = u''(0) = u'(1) = 0,\ u(0) = \mu. \end{aligned} \tag{43}$$

Obviously, due to the continuity of the functions f and g we have $Aw \in \mathcal{B}$. It is easy to verify the following.

Lemma 4.1. *If $w = (\varphi, \mu)^T$ is a fixed point of the operator A in the space $\mathcal{B}$, i.e., w is a solution of the operator equation*

$$Aw = w \tag{44}$$

in $\mathcal{B}$, then the function $u(t)$ defined from the problem (43) solves the original problem (41). Conversely, if $u(t)$ is a solution of (41), then the pair (φ, μ), where

$$\varphi(t) = f(t, u(t), u'(t), u''(t), u'''(t)), \tag{45}$$

$$\mu = \int_0^1 g(s)u(s)ds, \tag{46}$$

is a solution of the operator equation (44).

Thus, by this lemma, the problem (41) is reduced to the fixed point problem for A.

As in the case of ODEs in Section 2 via the study of the properties of the operator A we established the existence, uniqueness and positivity of solution of the integral BVP (41). Notice that our method for investigating IDE is completely different from the method of other authors such as Benaicha and Haddouchi in [106], Zhang and Ge [107] where the authors used the Green function of the IDE and the fixed point theory on cones.

After establishing the existence of unique solution we also proposed an iterative method on continuous level for finding the solution:

1. Given
$$\varphi_0(t) = f(t, 0, 0, 0, 0), \ \mu_0 = 0. \tag{47}$$
2. Knowing $\varphi_k(t)$ and μ_k $(k = 0, 1, ...)$ compute
$$\begin{aligned} u_k(t) &= \int_0^1 G_0(t,s)\varphi_k(s)ds + \mu_k, \\ y_k(t) &= \int_0^1 G_1(t,s)\varphi_k(s)ds, \\ v_k(t) &= \int_0^1 G_2(t,s)\varphi_k(s)ds, \\ z_k(t) &= \int_0^1 G_3(t,s)\varphi_k(s)ds. \end{aligned} \tag{48}$$
3. Update
$$\begin{aligned} \varphi_{k+1}(t) &= f(t, u_k(t), y_k(t), v_k(t), z_k(t)), \\ \mu_{k+1} &= \int_0^1 g(s)u_k(s)ds. \end{aligned} \tag{49}$$

This iterative method was discretized by using the trapezoidal quadrature formula and the total error is $O(h^2)$.

The technique used in [103] was developed for an problem with two integral boundary conditions in [105] and for a third order integral BVP in [104]. The construction of high order numerical methods for integral BVPs is needed and it is subject of our research in the future.

4.2. Integro-differential equations

The general methodology for investigating nonlinear BVPs of ODEs can be extended to nonlinear integro-differential equations. Below for illustration we give an example. Namely, we consider the problem [67]

$$\begin{aligned} u^{(4)}(x) &= f(x, u(x), u'(x), \int_0^1 k(x,t)u(t)dt), \\ u(0) = 0,\ &u(1) = 0,\ u''(0) = 0,\ u''(1) = 0, \end{aligned} \tag{50}$$

where the function $f(x, u, v, z)$ and $k(x, t)$ are assumed to be continuous. We introduce the operator A defined in the space of continuous functions $C[0, 1]$ by the formula

$$(A\varphi)(x) = f\Big(x, u(x), u'(x), \int_0^1 k(x,t)u(t)dt\Big), \tag{51}$$

where $u(x)$ is the solution of the boundary value problem

$$\begin{aligned} u'''' &= \varphi(x),\ 0 < x < 1, \\ u(0) = u''(0) &= u(1) = u''(1) = 0. \end{aligned} \tag{52}$$

Due to the following lemma the solution of the problem for IDE is reduced to finding fixed point of the operator A.

Lemma 4.2. *If the function φ is a fixed point of the operator A, i.e., φ is the solution of the operator equation*

$$A\varphi = \varphi, \tag{53}$$

where A is defined by (51)-(52) then the function $u(x)$ determined from the BVP (52) is a solution of the BVP (50). Conversely, if the function $u(x)$ is the solution of the BVP (50) then the function

$$\varphi(x) = f(x, u(x), u'(x), \int_0^1 k(x,t)u(t)dt)$$

satisfies the operator equation (53).

In [67] via the operator A we established the existence, uniqueness and positivity of solution. And importantly, we proposed an iterative method for finding the solution of (50) as follows:

1. Given
$$\varphi_0(x) = f(x, 0, 0, 0).$$

2. Knowing $\varphi_m(x)$ $(m = 0, 1, ...)$ compute
$$\begin{aligned} u_m(x) &= \int_0^1 G_0(x,t)\varphi_m(t)dt, \\ v_m(x) &= \int_0^1 G_1(x,t)\varphi_m(t)dt, \\ z_m(x) &= \int_0^1 k(x,t)u_m(t)dt. \end{aligned}$$

3. Update

$$\varphi_{m+1}(x) = f(x, u_m(x), v_m(x), z_m(x)).$$

Discretizing the above iterative method by use of trapezoidal formula we obtained a numerical method of accuracy $O(h^2)$. Of course, if using the trapezoidal formula with corrections as in Subsection 3.1 we can construct higher order numerical methods.

5. FUNCTIONAL DIFFERENTIAL EQUATIONS (FDEs)

In recent years functional differential equations have attracted attention of some authors since they provide realistic models for many phenomena and occupy a core and leading role in a wide spectrum of application areas in engineering and sciences, particularly in the biological sciences. Except for initial value problems, boundary value problems are of interest of researchers. In [75] the authors considered the problem

$$\begin{aligned} &u^{(2p)}(t) = f(t, u(t), u(\varphi(t))),\ t \in (a, b),\\ &u^{(i)}(a) = a_i, u^{(i)}(b) = b_i,\ i = \overline{0, p-1}, \end{aligned} \tag{54}$$

where $\varphi : [a, b] \to \mathbb{R}, a \le \varphi(t) \le b, \forall t \in [a, b]$. Neglecting the issue of existence and uniqueness of solutions the authors proposed an iterative method with use of a cubic spline interpolation procedure activated at each iteration. This indeed is the successive approximation method applied to the equivalent integral equation. Some error estimates were obtained for the cases $p = 1$ and $p = 2$ but in the proof of these estimates there were some vital errors relating to the derivatives of the Green function. These errors were overcome in corrigendum in [76].

Motivated by the work [75], in [77] we consider the FDE

$$\begin{aligned} &u''' = f(t, u(t), u(\varphi(t))), \quad 0 < t < a,\\ &B_1[u] = b_1, B_2[u] = b_2, B_3[u] = b_3, \end{aligned} \tag{55}$$

where $\varphi(t)$ is a continuous function mapping $[0, a]$ into itself, $B_1[u], B_2[u], B_3[u]$ are defined as in (3). Analogously as in the case of ODEs and IDEs we introduce the nonlinear operator A defined in the space of continuous functions $C[0, a]$ by the formula

$$(A\psi)(t) = f(t, u(t), u(\varphi(t))), \tag{56}$$

where $u(t)$ is the solution of the problem

$$\begin{aligned} &u'''(t) = \psi(t),\ 0 < t < a,\\ &B_1[u] = b_1, B_2[u] = b_2, B_3[u] = b_3. \end{aligned} \tag{57}$$

We also proved that the solution of the problem (55) is equivalent to the solution of the operator $A\psi = \psi$. It is due to this we obtained results on the existence, uniqueness of solution and proposed the following iterative method:

1. Given $\psi_0 \in B[0, M]$, for example,

$$\psi_0(t) = f(t, 0, 0). \tag{58}$$

2. Knowing $\psi_k(t)$ $(k = 0, 1, ...)$ compute

$$\begin{aligned} u_k(t) &= g(t) + \int_0^a G(t,s)\psi_k(s)ds, \\ v_k(t) &= g(\varphi(t)) + \int_0^a G(\varphi(t),s)\psi_k(s)ds. \end{aligned} \tag{59}$$

3. Update

$$\psi_{k+1}(t) = f(t, u_k(t), v_k(t)). \tag{60}$$

Above $G(t, s)$ is the Green function of the corresponding third order BVP, $g(t)$ is the polynomial of second degree satisfying the boundary conditions. Using the trapezoidal formula for computing the integrals in the above iterative method we obtained a discrete iterative method which was proved to have the accuracy $O(h^2)$.

Developing the technique of [77], in [108] considered the boundary value problem for fourth order nonlinear functional differential equation which contains all lower derivatives of proportional delay arguments

$$\begin{aligned} &u''''(t) = f(t, U(t)),\ 0 < t < 1, \\ &u(0) = a, u(1) = b, u'(0) = c, u'(1) = d, \end{aligned} \tag{61}$$

where

$$U(t) = (u(t), u(\varphi_0(t)), u'(t), u'(\varphi_1(t)), u''(t), u''(\varphi_2(t)), u'''(t), u'''(\varphi_3(t))), \tag{62}$$

and $f : [0, 1] \times \mathbb{R}^8 \to \mathbb{R}$ and $\varphi_i : [0, 1] \to [0, 1]$ $(i = \overline{0, 3})$ are continuous functions. We established the existence of a unique solution of the problem and constructed iterative methods on both continuous and discrete levels.

It should be said that our work [77] was the motivation for Bica and his colleage to use the technique of [75] for the third order FDE (55). In [109] the authors proved that the maximal order of convergence of the their method for the third order FDE is three. In the next work [110] Bica proved that the maximal order of convergence of the method for the fourth order FDE is four. Besides they expected that the order convergence is $O(h^5)$ for fifth order FDE if using quartic splines interpolation procedure combined with an $O(h^5)$ quadrature rule.

In our preprint [78] using trapezoidal formula with corrections for the integrals in (59) we constructed methods of $O(h^4), O(h^6)$ and $O(h^8)$ of accuracy for BVPs for third, fourth and fifth orders BVPs. From the results of experiments for solving the FDEs we observe that the achieved accuracy is somewhat worse than the accuracy for ODEs in [97, 100]. The reason of this may be in the computation of the values of the derivatives of $\psi(t)$ at the point $\xi_i = \varphi(t_i)$ not coinciding with grid points t_i.

6. PARTIAL DIFFERENTIAL EQUATIONS

The general methodology to the solution of nonlinear BVPs presented in Section 2 can be applied to multidimensional elliptic BVPs. In this section we review some works concerning the study of existence and numerical solution of nonlinear biharmonic and triharmonic BVPs. First we mention the work [79] of the problem

$$\begin{aligned} \Delta^2 u &= f(x, u, \Delta u) \quad \text{in } \Omega, \\ u = 0, \quad \Delta u &= 0 \quad \text{on } \Gamma, \end{aligned} \tag{63}$$

where Ω is a connected bounded domain in $\mathbb{R}^2$, with a smooth boundary Γ, Δ is the Laplace operator. The problem (63) describes the static deflection of an elastic bending plate with hinged edges rested on nonlinear foundation. The existence of the problem was studied by many authors by the method of upper and lower solutions and different methods of nonlinear functional analysis.

To investigate the problem (63) we also associated it with a fixed point problem of the operator A defined by

$$(A\varphi)(x) = f(x, u(x), \Delta u(x)),$$

where $u(x)$ is the solution of the problem

$$\begin{aligned} \Delta^2 u &= \varphi(x), \ x \in \Omega, \\ u = \Delta u &= 0, \ x \in \Gamma. \end{aligned}$$

Under some conditions imposed on the function $f(x, u, v)$ we established the existence of a unique fixed point of A, which corresponds a unique solution of the problem (63) The successive approximation of this fixed point generates the following iterative method for solving the problem (63):

1. Given a starting approximation $\varphi_0 \in B[0, M]$, for example,

$$\varphi_0(x) = f(x, 0, 0), \ x \in \Omega.$$

2. Knowing φ_k in Ω $(k = 0, 1, ...)$ solve consecutively two second order problems

$$\begin{aligned} \Delta v_k &= \varphi_k, \quad x \in \Omega, \\ v_k &= 0, \quad x \in \Gamma, \end{aligned}$$

$$\begin{aligned} \Delta u_k &= v_k, \quad x \in \Omega, \\ u_k &= 0, \quad x \in \Gamma. \end{aligned}$$

3. Update the new approximation

$$\varphi_{k+1} = f(x, u_k, v_k).$$

To numerically solve the BVPs for the Poisson equation on each iteration we constructed fourth order difference schemes. In result we obtained a numerical method of fourth order of accuracy which is better than the method of Wang in [111, 112].

Developing the above method, in [80] we considered the BVP for a nonlinear biharmonic equation of Kirchhoff type

$$\begin{aligned} &\Delta^2 u = M\Big(\int_\Omega |\nabla u|^2 dx\Big)\Delta u + f(x, u), \ x \in \Omega, \\ &u = 0, \ \Delta u = 0, \ x \in \Gamma, \end{aligned} \tag{64}$$

where Ω is a connected bounded domain in $\mathbb{R}^K$ $(K \geq 2)$ with a smooth boundary Γ, Δ is the Laplace operator, ∇u is the gradient of u, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $M : \mathbb{R}^+ \to \mathbb{R}$ are continuous functions. This problem describes the nonlinear static deflection of an elastic plate.

In a similar way as done in [79] we studied the problem via the finding the fixed point of the operator A in $C(\bar{\Omega})$ defined by

$$(A\varphi)(x) = M\Big(\int_\Omega |\nabla u|^2 dx\Big)\Delta u + f(x,u),$$

where $u(x)$ is a solution of the problem

$$\begin{aligned} \Delta^2 u &= \varphi(x),\ x \in \Omega,\\ u &= \Delta u = 0,\ x \in \Gamma. \end{aligned}$$

The finding the fixed point of A or the solution of the problem (64) is carried out by the iterative method:

i) Given a starting approximation $\varphi_0 \in B[O,N]$, for example,

$$\varphi_0(x) = f(x,0),\ x \in \Omega.$$

ii) Knowing φ_k $(k = 0,1,2,...)$ solve successively two second order problems

$$\begin{aligned} \Delta v_k &= \varphi_k,\ x \in \Omega,\\ v_k &= 0,\ x \in \Gamma,\\ \Delta u_k &= v_k,\ x \in \Omega,\\ u_k &= 0,\ x \in \Gamma. \end{aligned}$$

iii) Compute the new approximation

$$\varphi_{k+1}(x) = M\Big(\int_\Omega |\nabla u_k|^2 dx\Big)v_k + f(x,u_k).$$

Extending the technique for nonlinear biharmonic equations in combination with the iterative method for linear triharmonic equation [113], recently in [114] we considered the problem

$$\Delta^3 u = f(x,u,\Delta u,\Delta^2 u),\ x \in \Omega, \tag{65}$$

$$u = b_0,\ \frac{\partial u}{\partial \nu} = b_1,\ \Delta u = b_2,\ x \in \Gamma. \tag{66}$$

Notice that when the domain Ω is a rectangle in $\mathbb{R}^2$ and $b_0 = 0$ then the boundary $\Delta u = b_2$ is the same as the condition $\frac{\partial^2 u}{\partial \nu^2} = b_2$. Therefore, instead the boundary conditions (66) it is possible consider the Dirichlet boundary conditions

$$u = b_0,\ \frac{\partial u}{\partial \nu} = b_1,\ \frac{\partial^2 u}{\partial \nu^2} = b_2. \tag{67}$$

In order to reduce the problem (65)-(66) to an operator equation we reduced the space H of pairs of functions $\varphi \in C(\Omega)$ and $g \in C(\Gamma)$ and denote

$$Z = \begin{bmatrix} \varphi \\ g \end{bmatrix}. \tag{68}$$

In the space H define the operator A by

$$AZ = \begin{bmatrix} f(.,u,v,w) \\ g - \tau\left(\dfrac{\partial u}{\partial \nu} - b_1\right) \end{bmatrix}, \tag{69}$$

where u, v, w are the solutions of the problems

$$\begin{aligned} \Delta w &= \varphi, \ x \in \Omega, \\ w &= g, \ x \in \Gamma, \end{aligned} \tag{70}$$

$$\begin{aligned} \Delta v &= w, \ x \in \Omega, \\ v &= b_2, \ x \in \Gamma, \end{aligned} \tag{71}$$

$$\begin{aligned} \Delta u &= v, \ x \in \Omega, \\ u &= b_0, \ x \in \Gamma, \end{aligned} \tag{72}$$

and τ is a positive parameter. It was proved that the fixed point of the operator equation $AZ = Z$ generates the solution of the triharmonic problem and it can be found by the iterative method

$$\begin{aligned} Z_{k+1} &= AZ_k, \ k = 0, 1, ... \\ Z_0 & \text{ is given.} \end{aligned}$$

This iterative method is realized by the following iterative process:

i) Given an initial approximation φ_0, g_0, for example,

$$\varphi_0(x) = f(x, 0, 0, 0), \ x \in \Omega; \ g_0 = 0, \ x \in \Gamma.$$

ii) Knowing $\varphi_k, \ g_k \ (k = 0, 1, 2, ...)$ solve sequentially three second order problems

$$\begin{aligned} \Delta w_k &= \varphi_k, \ x \in \Omega, \\ w_k &= g_k, \ x \in \Gamma, \\ \Delta v_k &= w_k, \ x \in \Omega, \\ v_k &= b_2, \ x \in \Gamma, \\ \Delta u_k &= v_k, \ x \in \Omega, \\ u_k &= b_0, \ x \in \Gamma. \end{aligned}$$

iii) Calculate the new approximation

$$\begin{aligned} \varphi_{k+1}(x) &= f(x, u_k(x), v_k(x), w_k(x)), \\ g_{k+1} &= g_k - \tau\left(\frac{\partial u_k}{\partial \nu} - b_1\right). \end{aligned}$$

When the domain Ω is a rectangle, on each iteration discretizing the second order problems by difference schemes of fourth order of accuracy and computing the normal derivatives by difference formula also of the same order accuracy we obtain an approximate solution with convergence order of 4 although it was not proved theoretically. This result is much more accurate than that in [115].

7. DISCUSSION AND FUTURE RESEARCH DIRECTIONS

Some advantages of our approach

(i) The unified approach was first proposed for solving BVPs for ODEs and then was extended to IDEs, FDEs and PDEs. Its advantages over existing methods in investigation of qualitative aspects of BVPs such as the relaxing conditions for existence of solutions and the ease of verification of the conditions required. The approach is not only effective in the establishment of existence and uniqueness of solutions but also gives the way for constructing solution via iterations.

(ii) The construction of numerical methods for solving BVPs based on using trapezoidal formula with corrections for discretization of the integrals containing Green functions and their derivatives on each iteration of the continuous iterative methods gives high order accuracy of solutions and their derivatives. It essentially differs from the numerical methods of other authors based on discretization of differential equations and boundary conditions or integration of initial value problems obtained after using shooting methods.

(iii) The stability problem of our numerical methods does not arise because they are realization of the iterative methods for finding fixed points of operator equations.

(iv) The computational complexity of numerical algorithms for solving one-dimensional BVPs for ODEs, IDEs or FDEs on grid of N points is $O(KN^2)$, where K is the number of iterations performed, meanwhile the numerical algorithms of other authors, in general, require the computational cost $O(N^3)$ for solving the nonlinear system of N equations resulted from discretization of the BVPs.

(v) Although we considered only ODEs of order not higher than 6, which are mathematical models of many popular problems in physics and mechanics, the approach can be applied to higher order BVPs (see, e.g., [116, 117, 118]).

Limitations of the approach

As our approach is Green function based, it fails to apply to equations with variable coefficients, where it is difficult or even impossible to find Green functions, and is not directly applied to BVPs with nonlinear boundary conditions.

Future research directions

(i) Develop the approach in combination with other techniques to BVPs for ODEs associated with nonlinear boundary conditions. These problems arise, for example, in nonlinear composite beams [119]. Some of our initial results concerning this topic was obtained in [120], where we considered nonlinear beam equation subjected to nonlinear boundary moment conditions.

(ii) Extend the application of the unified approach to singular BVPs, which arise in the modelling of several phenomena in theoretical physics, astrophysics and chemistry. The typical problems of this type are Lane-Emden BVPs, for which there many numerical methods but their accuracy is not higher than 6.

(iii) Construction of highly accurate numerical methods for solving IDEs.

(iv) BVPs on infinite intervals also will be the subject of our research in the future.

8. CONCLUSION

In this overview we systematically presented a novel united approach to study nonlinear BVPs for ODEs including two-point and integral boundary condition problems, integro-differential, functional differential and partial differential equations. The common approach is to reduce the BVPs to operator equations for right-hand sides (or nonlinear terms) and use them to investigate the qualitative aspects of the problems and to construct iterative methods for finding the solutions. This approach allows to relax the conditions imposed on the nonlinear terms to ensure the existence of solutions. Thanks to this approach combined with the use of the trapezoidal quadrature formula with corrections, we have built high order numerical methods to solve BVPs for ODEs, and now we are constructing high order numerical methods for IDEs and FDEs. Also, combining this approach with the boundary operator method in [121] we established the existence and uniqueness of solutions for nonlinear biharmonic and triharmonic problems and found their solutions with fourth order of accuracy. The proposed approach can be applied to higher order equations of the three above types and higher order numerical methods can be constructed.

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