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ON SINGULAR HAMMERSTEIN EQUATIONS

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Abstract. In this paper the generalized solution of the singular Hammerstein equations in reflexive Banach spaces is defined and found by the operator method of regularization. Convergence rate of the regularized solution is studied. An application in mechanics is given for illustration.

1. INTRODUCTION

Let X be a real reflexive Banach space having E-property and X^* be dual space of X. For the sake of simplicity and without of any confresion norms of X and X^* will be denoted by the same symbol $\|.\|$. We write $\langle x^*, x\rangle$ instead of $x^*(x)$ for $x^* \in X^*$ and $x \in X$. Let F_i, $i = 1, 2$, be monotone, continuous and bounded operators with domain of definition $D(F_1) \subseteq X$, $D(F_2) \subseteq X^*$ and range $R(F_1) \subset D(F_2)$, $R(F_2) \subset X$.

Consider the operator equation of Hammerstein type

$$x + F_2F_1(x) = f, \ \ f \in X. \tag{1.1}$$

Equation (1.1) is called to be regular if $D(F_1) = X$ and $D(F_2) = X^*$, and singular otherwise. The existence of solutions of the regular equation (1.1) was studied in [2], [13] and [14]. The singular case of (1.1) was firstly investigated in [3], [4]. In this paper, basing on the results in the theory of system of variational inequalities (see [8]), the method of regularization (see [5]) and by introducing a new concept of solution for (1.1), called generalized solution, we give a method of approximating this solution when F_i are known approximatively by F_i^h.

Let G_i, $i = 1, 2$, be the convex and closed subsets of X and X^*, respectively, such that $G_1 \subseteq D(F_1)$, $G_2 \subseteq D(F_2)$, and $\operatorname{int} G_i \neq \emptyset$.

Definition. The element $x_0 \in G_1$ is called the generalized solution of (1.1), if there exists an element $x_0^* \in G_2$ such that

$$\langle F_1(x_0) - x_0^*, x - x_0\rangle \geq 0, \ \ \forall x \in G_1, \tag{1.2}$$

$$\langle F_2(x_0^*) + x_0 - f, x^* - x_0^*\rangle \geq 0, \ \ \forall x^* \in G_2, \tag{1.3}$$

It means that the pair $[x_0, x_0^*]$ is the solution of (1.2) and (1.3). If (1.1) is regular, then the classical solution x_0 is also the generalized solution with $x_0^* = F_1(x_0)$, $G_1 = X$ and $G_2 = X^*$, because x_0 and x_0^* satisfy the system of two equations

$$F_1(x) - x^* = 0, \ \ x + F_2(x^*) - f = 0. \tag{1.4}$$

Obviously, they satisfy (1.2) and (1.3). Inversely, if the generalized solution $x_0 \in \operatorname{int} G_1$ and $x_0^* \in \operatorname{int} G_2$, then x_0 is the classical solution, because x_0 and x_0^* satisfy the system (1.4) (see [12]). If both the sets G_i are bounded, the generalized solution always exists (see [1]). Moreover, if both the operators F_i are strictly monotone, then there exists a unique solution of (1.2) and (1.3), hence there exists a unique generalized solution of (1.1). Repeating entirely the proof of Theorems **X.1** and **X.2** in [12], we have two results on the existence of the generalized solution.

Theorem 1.1. *Let F_i be hemicontinuous monotone operators, and $G_1 \subseteq D(F_1)$ and $G_2 \subseteq D(F_2)$ be the convex and closed subsets. Also let $F_2(0) = 0$. Suppose that $\exists R > 0 : \langle F_1(u), u\rangle < 0 \to \|u\| \leq R$. Then there exists a generalized solution of (1.1).*

Theorem 1.2. *Suppose that $\exists R' > 0 : \langle F_1(x), x\rangle < 0 \to \|F_1(x)\| \leq R'$ and that F_2 is bounded with $F_2(0) = 0$. Then there exists a generalized solution of (1.1) ($\|F_1(x_0)\| \leq R'$).*

From now on, the symbols $\rightharpoonup$ and $\to$ denote weak convergence and convergence in norm, respectively.

2. MAIN RESULTS

Let U_1 be the standard dual mapping of X, i.e., U_1 is a mapping from X onto X^* having the property (see [14])

$$\langle U_1(x), x\rangle = \|U_1(x)\|\|x\| = \|x\|^2, \ \forall x \in X.$$

Also let $U_2 : X^* \to X$ be the standard dual mapping of X^*.

Consider the system of variational inequalities: find $x_\alpha \in G_1$, $x_\alpha^* \in G_2$ such that

$$\langle F_1(x_\alpha) + \alpha U_1(x_\alpha) - x_\alpha^*, x - x_\alpha\rangle \geq 0, \ \forall x \in G_1, \tag{2.1}$$

$$\langle F_2(x_\alpha^*) + \alpha U_2(x_\alpha^*) + x_\alpha - f, x^* - x_\alpha^*\rangle \geq 0, \ \forall x^* \in G_2, \tag{2.2}$$

where α is a small parameter. We have the following result.

Theorem 2.1. *For each $\alpha > 0$ the system of inequalities (2.1) and (2.2) has a unique solution $[x_\alpha, x_\alpha^*]$. Moreover, the sequence $\{x_\alpha\}$ converges to a generalized solution of (1.1), as $\alpha \to 0$.*

If instead of F_i it is only known the monotone hemicontinuous approximations F_i^h such that

$$\|F_1(x) - F_1^h(x)\| \leq hg_1(\|x\|), \ x \in G_1,$$

$$\|F_2(x^*) - F_2^h(x^*)\| \leq hg_2(\|x^*\|), \ x^* \in G_2, \ h \to 0,$$

$$g_i(t) \leq a_i + b_i t,$$

where $g_i(t)$ are the real and nondecreasing functions with $g_i(0) = 0$, $g_i(t) \to +\infty$, as $t \to +\infty$, then we can define a regularized solution as the solutions of the variational inequalities: find $x_{h\alpha} \in G_1$, $x_{h\alpha}^* \in G_2$ such that

$$\langle F_1^h(x_{h\alpha}) + \alpha U_1(x_{h\alpha}) - x_{h\alpha}^*, x - x_{h\alpha}\rangle \geq 0, \ \forall x \in G_1, \tag{2.3}$$

$$\langle F_2^h(x_{h\alpha}^*) + \alpha U_2(x_{h\alpha}) + x_{h\alpha} - f, x^* - x_{h\alpha}^*\rangle \geq 0, \ \forall x^* \in G_2. \tag{2.4}$$

Theorem 2.2. *For each $\alpha > 0$ system (2.3), (2.4) has a unique solution $[x_{h\alpha}, x_{h\alpha}^*]$, and if $h/\alpha \to 0$, the sequence $\{x_{h\alpha}\}$ converges to a generalized solution of (1.1).*

If X and X^* are uniformly convex, the solutions x_α or $x_{\alpha h}$ can be found by the methods in [8], because the operators $F_{i\alpha} = F_i + \alpha U_i$, $F_{i\alpha}^h = F_i^h + \alpha U_i$, $i = 1, 2$, for each $\alpha > 0$, are uniformly monotone. Our next result in this paper is concerned with the convergence rates of the sequences $\{x_\alpha\}$ and $\{x_{h\alpha}\}$.

Let the mappings U_i satisfy the following conditions

$$\langle U_i(y_1^i) - U_i(y_2^i), y_1^i - y_2^i\rangle \geq m_i\|y_1^i - y_2^i\|^{s_i}, \ \ m_i > 0, \ s_i \geq 2, \tag{2.5}$$

$$\|U_i(y_1^i) - U_i(y_2^i)\| \leq c_i(R_i)\|y_1^i - y_2^i\|^{\nu_i}, \ \ 0 < \nu_i \leq 1, \tag{2.6}$$

where y_1^i, $y_2^i \in X$ or X^* on dependence of $i = 1$ or 2, respectively, and $c_i(R_i)$, $R_i > 0$, are the positive increasing functions on $R_i = \max\{\|y_1^i\|, \|y_2^i\|\}$ (see [11]).

Assume that x_0 is a solution in the classical sense of the equation (1.1).

Theorem 2.3. *Suppose that the following conditions hold:*

(i) F_1 is Fréchet differentiable at some neighborhood $\mathcal{U}_0$ of x_0 $s_1 - 1$-times if $s_1 = [s_1]$, the integer part of s_1, $[s_1]$-times if $s_1 \neq [s_1]$, and F_2 is Fréchet differentiable at some neighborhood $\mathcal{V}_0$ of x_0^* $s_2 - 1$-times, if $s_2 = [s_2]$, $[s_2]$-times if $s_2 \neq [s_2]$,

(ii) there exists a constant $\tilde{L} > 0$ such that

$$\|F_1^{(k)}(x_0) - F_1^{(k)}(y)\| \leq \tilde{L}\|x_0 - y\|, \ \forall y \in \mathcal{U}_0,$$
$$\|F_2^{(k)}(x_0^*) - F_2^{(k)}(y^*)\| \leq \tilde{L}\|x_0^* - y^*\|, \ \forall y^* \in \mathcal{V}_0,$$

for $F_i^{(k)}$: $k = s_i - 1$ if $s_i = [s_i]$, $k = [s_i]$ if $s_i \neq [s_i]$, and if $[s_i] \geq 3$, then $F_1^{(2)}(x_0) = \ldots = F_1^{(k)}(x_0) = 0$, and $F_2^{(2)}(x_0^*) = \cdots = F_1^{(k)}(x_0^*) = 0$,

(iii) there exists an element $x^1 \in X$ such that

$$(I + F_2'(x_0^*)^* F_1'(x_0)^*)x^1 = F_2'(x_0^*)^* U_1(x_0) - U_2(x_0^*),$$

if $s_1 = [s_1]$ then $\tilde{L}\|x^1\| < m_1 s_1!$, and if $s_2 = [s_2]$ then $\tilde{L}\|F_1'(x_0)^* x^1 - U_1(x_0)\| < m_2 s_2!$
Then, if α is chosen such that $\alpha \sim h^\rho$, $0 < \rho < 1$, then

$$\|x_{h\alpha} - x_0\| \leq O(h^{\theta/s_1}), \ \theta = \min\{\rho, 1-\rho\}.$$

3. PROOFS

Following [5], consider the Banach space $Z = X \times X^*$ with the norm of any element $z \in Z$, $z = [x, x^*]$, $x \in X$, $x^* \in X^*$ defined by

$$\|z\| = (\|x\|^2 + \|x^*\|^2)^{1/2}.$$

Then, the system of equations (1.4) can be written in the form

$$\mathcal{F}(z) = \overline{f}, \tag{3.1}$$

where $\mathcal{F}(z) = [F_1(x) - x^*, x + F_2(x^*)]$, $\overline{f} = [0, f]$. It is easy to see that $\mathcal{F}$ is a monotone operator from Z into $Z^* = X^* \times X$. Analogously, the systems of variational inequalities (1.2), (1.3) and (2.1), (2.2) can be written in the form

$$\langle \mathcal{F}(z_0) - \overline{f}, z - z_0 \rangle \geq 0, \ \forall z \in G, \tag{3.2}$$
$$\langle \mathcal{F}(z_\alpha) + \alpha J(z_\alpha) - \overline{f}, z - z_\alpha \rangle \geq 0, \ \forall z \in G, \tag{3.3}$$

respectively, where $G == G_1 \times G_2$, $z_0 = [x_0, x_0^*]$, $z_\alpha = [x_\alpha, x_\alpha^*]$, and J is the standard dual mapping of the space Z. From the results in the theory of regularization for variational inequalities (see [10]) we obtain that (3.3) has a unique solution $z_\alpha := [x_\alpha, x_\alpha^*]$, $\|x_\alpha\| \to \|x_0\|$, $x_\alpha \rightharpoonup x_0$, as $\alpha \to 0$. Because X possesses E-property, the sequence $\{x_\alpha\}$ converges strongly to x_0. Theorem 2.1 is proved.

We rewrite the system (2.3) and (2.4) in the form

$$\langle \mathcal{F}^h(z_{h\alpha}) + \alpha J(z_{h\alpha}) - \overline{f}, z - z_{h\alpha} \rangle \geq 0, \ \forall z \in G,$$

where $z_{h\alpha} = [x_{h\alpha}, x_{h\alpha}^*]$, $\mathcal{F}^h(z) = [F_1^h(x) - x^*, x + F_2^h(x^*)]$ and is monotone. It is easy to verify that

$$\|\mathcal{F}^h(z) - \mathcal{F}(z)\| \leq hg(\|z\|), \ z \in G,$$

where $g(t) = \max\{g_1(t), g_2(t)\}$ and it has all the same properties as $g_i(t)$ do. If $h/\alpha \to 0$, then $x_{h\alpha} \to x_0$, as $\alpha \to 0$ (see [7]). Theorem 2.2 is proved.

Put

$$c = m_1\|x_{h\alpha} - x_0\|^{s_1} + m_2\|x^*_{h\alpha} - x^*_0\|^{s_2}.$$

Basing on (2.3) - (2.6) we have got

$$\begin{aligned} c \leq &\langle U_1(x_0), x_0 - x_{h\alpha}\rangle + \langle U_2(x^*_0), x^*_0 - x^*_{h\alpha}\rangle \\ &+ \frac{1}{\alpha}\Big[\langle x^*_{h\alpha} - F^h_1(x_{h\alpha}), x_{h\alpha} - x_0\rangle + \langle f - x_{h\alpha} - F^h_2(x^*_{h\alpha}), x^*_{h\alpha} - x^*_0\rangle\Big]. \end{aligned} \tag{3.4}$$

Put $x^2 = U_1(x_0) - F'_1(x_0)^*x^1$. From condition (iii) of Theorem 2.3 it follows that x^1 and x^2 ($\in X^*$) satisfy the system of following equalities

$$\begin{aligned} F'_1(x_0)^*x^1 + x^2 &= U_1(x_0), \\ F'_2(x^*_0)^*x^2 - x^1 &= U_2(x^*_0). \end{aligned}$$

Therefore, from (3.4), the monotone property of F_i and that x_0 is a solution of (1.1) in the classical sense, it implies that

$$\begin{aligned} c \leq &\langle U_1(x_0), x_0 - x_{h\alpha}\rangle + \langle U_2(x^*_0), x^*_0 - x^*_\alpha\rangle \\ &+ \frac{1}{\alpha}\Big[\langle x^*_{h\alpha} - x^*_0, x_{h\alpha} - x_0\rangle + \langle x_0 - x_{h\alpha}, x^*_{h\alpha} - x^*_0\rangle \\ &+ \langle F_1(x_0) - F^h_1(x_{h\alpha}), x_{h\alpha} - x_0\rangle + \langle F_2(x^*_0) - F^h_2(x^*_{h\alpha}), x^*_{h\alpha} - x^*_0\rangle\Big] \\ \leq &\langle U_1(x_0), x_0 - x_{h\alpha}\rangle + \langle U_2(x^*_0), x^*_0 - x^*_\alpha\rangle \\ &+ \frac{1}{\alpha}\Big[\langle x^*_{h\alpha} - x^*_0, x_{h\alpha} - x_0\rangle + \langle x_0 - x_{h\alpha}, x^*_{h\alpha} - x^*_0\rangle \\ &+ \langle F_1(x_0) - F_1(x_{h\alpha}), x_{h\alpha} - x_0\rangle + \langle F_2(x^*_0) - F_2(x^*_{h\alpha}), x^*_{h\alpha} - x^*_0\rangle\Big] \\ &+ \frac{Ch}{\alpha}(\|x_{h\alpha} - x_0\| + \|x^*_{h\alpha} - x^*_0\|) \\ \leq &\frac{Ch}{\alpha}(\|x_{h\alpha} - x_0\| + \|x^*_{h\alpha} - x^*_0\|) \\ &+ \langle x^2, x_0 - x_{h\alpha}\rangle + \langle x^1, F'_1(x_0)(x_0 - x_{h\alpha})\rangle \\ &+ \langle -x^1, x^*_0 - x^*_{h\alpha}\rangle + \langle x^2, F'_2(x^*_0)(x^*_0 - x^*_{h\alpha})\rangle, \end{aligned} \tag{3.5}$$

where C is some positive constant such that $g_1(\|x_{h\alpha}\|), g_2(\|x^*_{h\alpha}\|) \leq C$. First, consider the case $s_i = [s_i]$, $i = 1, 2$. As

$$\begin{aligned} F'_1(x_0)(x_0 - x_{h\alpha}) &= F_1(x_0) - F_1(x_{h\alpha}) + r_{h\alpha}, \\ F'_2(x^*_0)(x^*_0 - x^*_\alpha) &= F_2(x^*_0) - F_2(x^*_{h\alpha}) + \tilde{r}_{h\alpha}, \end{aligned}$$

$$\|r_{h\alpha}\| \leq \frac{\tilde{L}}{s_1!}\|x_{h\alpha} - x_0\|^{s_1}, \quad \|\tilde{r}_{h\alpha}\| \leq \frac{\tilde{L}}{s_2!}\|x^*_{h\alpha} - x^*_0\|^{s_2},$$

form the inequality (3.5) it follows

$$\begin{aligned} c \leq &\frac{Ch}{\alpha}(\|x_{h\alpha} - x_0\| + \|x^*_{h\alpha} - x^*_0\|) \\ &+ \langle x^2, x_0 - x_{h\alpha}\rangle + \langle -x^1, x^*_0 - x^*_{h\alpha}\rangle + \langle x^1, F_1(x_0) - F_1(x_{h\alpha})\rangle \\ &+ \langle x^2, F_2(x^*_0) - F_2(x^*_{h\alpha})\rangle + \frac{\tilde{L}\|x^1\|}{s_1!}\|x_{h\alpha} - x_0\|^{s_1} \\ &+ \frac{\tilde{L}\|x^2\|}{s_2!}\|x^*_{h\alpha} - x^*_0\|^{s_2} \end{aligned}$$

$$\begin{aligned}
&\le \frac{Ch}{\alpha}\left(\|x_{h\alpha} - x_0\| + \|x^*_{h\alpha} - x^*_0\|\right)\\
&\quad + \langle x^1, x^*_{h\alpha} - F_1(x_{h\alpha})\rangle + \langle x^2, f - x_{h\alpha} - F_2(x^*_{h\alpha})\rangle\\
&\quad + \frac{\tilde{L}\|x^1\|}{s_1!}\|x_{h\alpha} - x_0\|^{s_1} + \frac{\tilde{L}\|x^2\|}{s_2!}\|x^*_{h\alpha} - x^*_0\|^{s_2}\\
&\le \frac{Ch}{\alpha}\left(\|x_{h\alpha} - x_0\| + \|x^*_{h\alpha} - x^*_0\|\right)\\
&\quad + \alpha\langle x^1, U_1(x_{h\alpha})\rangle + \alpha\langle x^2, U_2(x^*_{h\alpha})\rangle + \frac{\tilde{L}\|x^1\|}{s_1!}\|x_{h\alpha} - x_0\|^{s_1}\\
&\quad + \frac{\tilde{L}\|x^2\|}{s_2!}\|x^*_{h\alpha} - x^*_0\|^{s_2}\\
&\le \frac{Ch}{\alpha}\left(\|x_{h\alpha} - x_0\| + \|x^*_{h\alpha} - x^*_0\|\right)\\
&\quad + \alpha\left(\|x^1\|\|x_{h\alpha}\| + \|x^2\|\|x^*_{h\alpha}\|\right) + \frac{\tilde{L}\|x^1\|}{s_1!}\|x_{h\alpha} - x_0\|^{s_1}\\
&\quad + \frac{\tilde{L}\|x^2\|}{s_2!}\|x^*_{h\alpha} - x^*_0\|^{s_2}.
\end{aligned}$$

Hence,

$$m_1\left(1 - \frac{\tilde{L}\|x^1\|}{m_1 s_1!}\right)\|x_{h\alpha} - x_0\|^{s_1} \le O(h^\rho + h^{1-\rho}). \tag{3.6}$$

Consequently,

$$\|x_{h\alpha} - x_0\| \le O(h^{\theta/s_1}).$$

If $s_i \ne [s_i]$ for one or both the two numbers s_i, for example $s_1 \ne [s_1]$, then

$$\|r_{h\alpha}\| \le \frac{\tilde{L}}{([s_1]+1)!}\|x_{h\alpha} - x_0\|^{[s_1]+1}$$

and the left-hand side of (3.6) will be replaced by

$$m_1\left(1 - \frac{\tilde{L}\|x^1\|}{m_1([s_1]+1)!}\|x_{h\alpha} - x_0\|^{[s_1]+1-s_1}\right)\|x_{h\alpha} - x_0\|^{s_1}.$$

Because $\|x_{h\alpha} - x_0\| \to 0$, and $[s_1] + 1 - s_1 > 0$, then

$$1 - \frac{\tilde{L}\|x^1\|}{([s_1]+1)!}\|x_{h\alpha} - x_0\|^{[s_1]+1-s_1} \ge 1/2$$

for sufficiently small α. The case $s_2 \ne [s_2]$ and both of the two numbers s_1, s_2 are not integer is considered analogously. This remark completes the proof of Theorem 2.3.

4. APPLICATION

Consider here an annular elastic membrane under the action of axisymmetric surface loads and uniform radial edge stresses or displacements within the Foppl-Henky theory. This leads us to consider the nonlinear differential equation

$$y'' + 3y'/x + 2R^2(x)/y^2 = 0, \quad 0 < a < x < 1, \tag{4.1}$$

with different kind of boundary conditions, where $R(x)$ is nondecreasing with $R(a) = 0$. By means of appropriate Green function (see [6]), the problem can be written as integral equation of the following form

$$y(x) = f(x) - \int_0^1 k(x,t)g(t, y(t))\,dt,$$

where the following conditions are fulfilled:

(i) $g: [0,1] \times [\varepsilon, \infty) \to R$ is Lipschitz continuous for every $\varepsilon > 0$;

(ii) $g(t,.) : R^+ \to R$ is nondecreasing for each $t \in [0,1]$;

(iii) The kernel $k(x,t)$ is continuous, symmetric, and positive semidefined, i.e.,

$$\int_0^1 \int_0^1 k(x,t)h(x)h(t)dxdt \geq 0, \ \forall h \in L_2[0,1];$$

(iv) f is continuous.

To apply our above theoretical results, we take $X = X^* = L_2[0,1]$,

$$(F_1\varphi)(t) = g(t, \varphi(t)), \ \varphi(t) \geq 0, \ a.e., \varphi(t) \in L_2[0,1],$$

$$(F_2h)(t) = \int_0^1 k(t,s)h(s)ds, \ h(s) \in L_2[0,1].$$

Therefore, $G_2 = L_2[0,1]$, $G_1 = \{\varphi \in L_2[0,1], \ \varphi(t) \geq 0, \ a.e.\}$. For the case $R^2 \equiv 1$, and the boundary conditions $y'(0) = 0$, $y(1) = S > 0$ the classical positive solution exists uniquely (see [9]).

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