ITERATIVE METHOD FOR SOLVING THE SECOND BOUNDARY VALUE PROBLEM (BVP) FOR BIHARMONIC TYPE EQUATION*

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Abstract. Solving BVPs for the fourth order differential equations by the reduction of them to BVPs for the second order equations with the aim to use the achievements for the latter ones attracts attention from many researchers.

In this paper, using the technique developed by ourselves in recent works, we construct iterative method for the second BVP for biharmonic type equation. The convergence rate of the method is established.

1. INTRODUCTION

Solving BVPs for the fourth order differential equations by the reduction of them to BVPs for the second order equations with the aim to use a lot of efficient algorithms for the latter ones attracts attention from many researchers. Namely, for the biharmonic equation with the Dirichlet boundary condition, there is intensively developed the iterative method, which leads the problem to two problems for the Poisson equation at each iteration (see e.g. [3, 8, 9, 11]).

Recently, Abramov and Ulijanova [1] proposed an iterative method for the Dirichlet problem for the biharmonic type equation, but the convergence of the method is not proved. In our previous works [5, 6] with the help of boundary or mixed boundary-domain operators appropriately introduced, we constructed iterative methods for biharmonic and biharmonic type equations associated with the Dirichlet boundary condition. It is proved that the methods are convergent with the rate of geometric progression.

In this paper we develop our technique in [4 - 6] for the second BVP for the biharmonic type equation. Namely, we consider the following problem.

$$\Delta^2 u - a\Delta u + bu = f(x), \quad x \in \Omega, \tag{1.1}$$

$$\Delta u|_\Gamma = g, \tag{1.2}$$

$$u|_\Gamma = h, \tag{1.3}$$

where Δ is the Laplace operator, Ω is a bounded domain in R^n $(n \geq 2)$, Γ is the sufficiently smooth boundary of Ω, a and b are constants. This problem has not yet considered in [1].

It should be noted that when $a = 0$, $b = 0$ Problem (1.1)–(1.3) is decomposed into two consecutive problems for the Poisson equation. Otherwise, in the case, if one of the two coefficients a and b is negative, as mentioned in [2], Problem (1.1)–(1.3) is approximated by a 13-point difference scheme and a module for the latter problem is constructed. In [7] the problem in a rectangle, where $a = 0$, $b > 0$, is reduced to a system of two boundary integral equations and then is solved by the use of the Galerkin and collocation methods.

The present work is devoted to the solution of Problem (1.1)–(1.3), where $a \geq 0$, $b \geq 0$ and $a + b > 0$. Two different cases will be treated in dependence on the sign of $a^2 - 4b$.

2. CASE $a^2 - 4b \geq 0$

In this case we always can lead the original problem (1.1)–(1.3) to two problems for Helhomn equation. To do this, as in [5], we put

$$\mu = \frac{1}{2}\left(a + \sqrt{a^2 - 4b}\right).$$

Then Problem (1.1)–(1.3) is reduced to the following problems

$$\mu \Delta v - bv = f, \quad x \in \Omega,$$

$$v|_\Gamma = \frac{1}{\mu}g - h,$$

$$\frac{1}{\mu}\Delta u - u = v, \quad x \in \Omega,$$

$$u|_\Gamma = h.$$

3. CASE $a^2 - 4b < 0$

This case is very important in mechanics, especially when $a = 0$, $b > 0$, because then we have the governing equation of a plate on elastic foundation (see [14]).

3.1. Construction of iterative method

As in [6], we set

$$\Delta u = v, \tag{3.1}$$

$$D = -bu. \tag{3.2}$$

Then Problem (1.1)–(1.3) is led to the following two second order problems

$$\Delta v - av = f + D, \quad x \in \Omega,$$
$$v|_\Gamma = g, \tag{3.3}$$

$$\Delta u = v, \quad x \in \Omega,$$
$$u|_\Gamma = h, \tag{3.4}$$

where all the functions u, v and D are unknown but they are related with each other by (3.2).

To solve the above problems we use the following iterative process:

Step 1. Given a starting approximation $D^{(0)}$.

Step 2. Knowing $D^{(k)}$ $(k = 0, 1, ...)$ solve consecutively two problems

$$\Delta v^{(k)} - av^{(k)} = f + D^{(k)}, \quad x \in \Omega,$$
$$v^{(k)}|_\Gamma = g, \tag{3.5}$$

$$\Delta u^{(k)} = v^{(k)}, \quad x \in \Omega,$$
$$u^{(k)}|_\Gamma = h. \tag{3.6}$$

Step 3. Compute the new approximation

$$D^{(k+1)} = D^{(k)} - \tau_{k+1}\left(D^{(k)} + bu^{(k)}\right), \quad x \in \Omega, \tag{3.7}$$

where τ_{k+1} are iterative parameters to be determined afterwards.

3.2. Investigation of convergence

In order to investigate the convergence of the iterative process (3.5)–(3.7), firstly we rewrite (3.7) in the canonical form of two-layer iterative scheme [12, 13]

$$\frac{D^{(k+1)} - D^{(k)}}{\tau_{k+1}} + \left(D^{(k)} + bu^{(k)}\right) = 0. \tag{3.8}$$

Now, we introduce the operator B defined by the formula

$$BD = D + bu, \tag{3.9}$$

where u is the function determined from the problems

$$\begin{aligned} \Delta v - av &= D, \quad x \in \Omega, \\ v|_\Gamma &= 0, \end{aligned} \tag{3.10}$$

$$\begin{aligned} \Delta u &= v, \quad x \in \Omega \\ u|_\Gamma &= 0, \end{aligned} \tag{3.11}$$

The properties of the operator B will be investigated in the sequel.

Next, in Problems (3.3) and (3.4) we put

$$u = u_1 + u_2, \quad v = v_1 + v_2,$$

where u_1, u_2, v_1, v_2 are the solutions of the problems

$$\Delta v_1 - av_1 = D, \quad x \in \Omega, \quad v_1|_\Gamma = 0, \tag{3.12}$$

$$\Delta u_1 = v_1, \quad x \in \Omega, \quad u_1|_\Gamma = 0, \tag{3.13}$$

$$\Delta v_2 - av_2 = f, \quad x \in \Omega, \quad v_2|_\Gamma = g, \tag{3.14}$$

$$\Delta u_2 = v_2, \quad x \in \Omega, \quad u_2|_\Gamma = h. \tag{3.15}$$

From (3.12) and (3.13) by the definition of B we have

$$BD = D + bu_1. \tag{3.16}$$

We also see that (3.14) and (3.15) define an unique function u_2.

Since the solution u found (3.3) and (3.4) should satisfy the relation (3.2), we have

$$D + b(u_1 + u_2) = 0. \tag{3.17}$$

Hence, from (3.16) it follows

$$BD = F, \tag{3.18}$$

where

$$F = -bu_2. \tag{3.19}$$

We see that (3.18) is an operator equation for D, where the right-hand side F is completely defined by the given functions f, g, h of Problem (1.1)–(1.3).

Proposition 1. *The iterative process (3.5)–(3.7) is a realization of the two-layer iterative scheme*

$$\frac{D^{(k+1)} - D^{(k)}}{\tau_{k+1}} + BD^{(k)} = F, \quad k = 0, 1, 2, \ldots \tag{3.20}$$

for the operator equation (3.18).

Proof. Indeed, if in (3.5)–(3.7) we put

$$u^{(k)} = u_1^{(k)} + u_2, \quad v^{(k)} = v_1^{(k)} + v_2, \tag{3.21}$$

where u_2, v_2 are solutions of Problems (3.14), (3.15), then we get

$$\begin{aligned} \Delta v_1^{(k)} - a v_1^{(k)} &= D^{(k)}, \quad x \in \Omega, \\ v_1^{(k)}|_\Gamma &= 0, \end{aligned} \tag{3.22}$$

$$\begin{aligned} \Delta u_1^{(k)} &= v_1^{(k)}, \quad x \in \Omega, \\ u_1^{(k)}|_\Gamma &= 0. \end{aligned} \tag{3.23}$$

From here it is easy to see that

$$BD^{(k)} = D^{(k)} + b u_1^{(k)}. \tag{3.24}$$

Taking into account (3.19) we obtain

$$D^{(k)} + b u^{(k)} = D^{(k)} + b u_1^{(k)} + b u_2^{(k)} = BD^{(k)} - F.$$

Consequently, the scheme (3.8) may be written in the form of (3.20).
Thus, the proposition is proved.

Proposition 1 enables us to lead the investigation of convergence of Process (3.5)–(3.7) to the study of the scheme (3.20).
For this reason we need some properties of the operator B.

Proposition 2. *The operator B defined by (3.9)–(3.11) is linear, symmetric positive definite and bounded in the space $L_2(\Omega)$ of quadratic integralable functions in Ω.*

Proof. The linearity of B is obvious. To establish the other properties of B let us consider the inner product $(BD, \overline{D})$ of two arbitrary functions D and $\overline{D}$ in $L_2(\Omega)$. We have

$$(BD, \overline{D}) = (D + bu, \overline{D}) = (D, \overline{D}) + b \int_\Omega u \overline{D} \, dx. \tag{3.25}$$

Here u is found from (3.10)–(3.11). We also denoted by $\overline{u}$ and $\overline{v}$ the solution of (3.10)–(3.11), where D is replaced by $\overline{D}$.

Now, we compute the integral in right-hand side of (3.25), which is denoted by J. We have

$$J = \int_\Omega u \overline{D} \, dx = \int_\Omega u (\Delta \overline{v} - a \overline{v}) \, dx = \int_\Omega u \Delta \overline{v} \, dx - \int_\Omega a u \overline{v} \, dx$$

$$= \int_\Omega u \Delta \overline{v} \, dx - a \int_\Omega u \Delta \overline{u} \, dx.$$

Applying the Green formula for functions vanishing on boundary, we obtain

$$J = \int_\Omega (v\bar{v} + a\,\mathrm{grad}\,u.\mathrm{grad}\,\bar{u})\,dx.$$

Hence,

$$(BD, \overline{D}) = \int_\Omega (D\overline{D} + v\bar{v} + a\,\mathrm{grad}\,u.\mathrm{grad}\,\bar{u})\,dx.$$

Clearly,

$$(BD, \overline{D}) = (B, \overline{D}, D)$$

and

$$(BD, D = \int_\Omega (D^2 + v^2 + a\,|\mathrm{grad}\,u|^2)\,dx \geq (D, D).$$

Thus, B is symmetric and positive definite in $L_2(\Omega)$.

Further, from the theory of BVPs in [10], for the solutions u and v of Problems (3.10), (3.11) we have the estimates

$$\|v\|_{H^2(\Omega)} \leq \sqrt{C_1}\,\|D\|_{L_2(\Omega)} \tag{3.26}$$

$$\|u\|_{H^2(\Omega)} \leq \sqrt{C_2}\,\|v\|^v_{L_2(\Omega)} \tag{3.27}$$

where C_1 and C_2 are positive constants, $H^s(\Omega)$ is the Sobolev space.

Using these estimates it is easy to get the following estimate

$$(BD, D) \leq (1 + C_2 + a\,C_1 C_2)(D, D), \tag{3.28}$$

which implies the boundness of B in $L_2(\Omega)$.

Thus, the proof of Proposition 2 in complete.

In future, besides (3.26), (3.27) we shall need the following estimate for the solution (3.11)

$$\|u\|_{H^4(\Omega)} \leq C_3\,\|v\|_{H^2(\Omega)}. \tag{3.29}$$

Now, we are able to state the result of convergence of the iterative process, constructed in the subsection 3.1.

Below, we assume that $f \in L_2(\Omega)$, $g \in H^{3/2}(\Gamma)$ and $h \in H^{7/2}(\Gamma)$. Then BVP (1.1)–(1.3) has an unique solution $u \in H^4(\Omega)$ (see [10]). Here, it is recalled that $H^s(\Omega)$ and $H^s(\Gamma)$ are the Sobolev spaces.

Theorem 1. *Let γ_1 and γ_2 be the lower and upper bounds of the spectrum of the operator B, i.e.*

$$\gamma_1 I \leq B \leq \gamma_2 I, \tag{3.30}$$

where I is the identity operator in $L_2(\Omega)$.

Let u^* be the solution of Problem (1.1)–(1.3) and D^* be the solution of Equation (3.18).

Then, if $\{\tau_{k+1}\}$ is the Chebyshev collection of parameters, constructed by the bounds γ_1 and γ_2, namely

$$\tau_0 = \frac{2}{\gamma_1 + \gamma_2}, \quad \tau_k = \frac{\tau_0}{\rho_0\, t_k + 1}, \quad t_k = \cos\frac{2k-1}{2M}\pi, \quad k = 1, 2, ..., M \tag{3.31}$$

$$\rho_0 = \frac{1-\xi}{1+\xi}, \quad \xi = \frac{\gamma_1}{\gamma_2}$$

we have

$$\left\| u^{(M)} - u^* \right\|_{H^4(\Omega)} \leq C\, q_M \,, \tag{3.32}$$

where

$$C = \sqrt{C_1}\, C_3 \left\| D^{(0)} - D^* \right\|_{L_2(\Omega)} \,, \tag{3.33}$$

C_1, C_3 *being the constants in (3.26), (3.29), respectively*

$$q_M = \frac{2\rho_1{}^M}{1 + \rho_1{}^{2M}}\,, \quad \rho_1 = \frac{1 - \sqrt{\xi}}{1 + \sqrt{\xi}}\,. \tag{3.34}$$

In the case of the stationary iterative process, $\tau_k = \tau_0$ $(k = 1, 2, ...)$ *we have*

$$\left\| u^{(k)} - u^* \right\|_{H^4(\Omega)} \leq \rho_0{}^k \left\| D^{(0)} - D^* \right\|_{L_2(\Omega)} \,, \quad k = 1, 2, ..., \tag{3.35}$$

Proof. According to the general theory of the two-layer iterative schemes (see [12, 13]) for the operator equation (3.18) we have

$$\left\| D^{(M)} - D^* \right\|_{L_2(\Omega)} \leq q_M \left\| D^{(0)} - D^* \right\|_{L_2(\Omega)} \,,$$

if the parameter $\{\tau_{k+1}\}$ is chosen by the formula (3.31) and

$$\left\| D^{(k)} - D^* \right\|_{H^4(\Omega)} \leq \rho_0{}^k \left\| D^{(0)} - D^* \right\|_{L_2(\Omega)} \,, \quad k = 1, 2, ...$$

if $\tau_k = \tau_0$ $(k = 1, 2, ...)$.

The corresponding estimates (3.32) and (3.35) follow from (3.26), (3.29) applied to the problems

$$\Delta\big(v^{(k)} - v^*\big) - a\big(v^{(k)} - v^*\big) = D^{(k)} - D^*, \quad x \in \Omega \,,$$

$$\big(v^{(k)} - v^*\big)\big|_\Gamma = 0 \,,$$

$$\Delta\big(u^{(k)} - u^*\big) = v^{(k)} - v^*, \quad x \in \Omega \,,$$

$$\big(u^{(k)} - u^*\big)\big|_\Gamma = 0 \,,$$

which are obtained in the result of the subtraction (3.3) and (3.4) from (3.5) and (3.6), respectively. Here, u^*, v^* and D^* satisfy (3.3) and (3.4).

4. CONCLUDING REMARK

The determination of the lower and upper bounds γ_1 and γ_2 of the spectrum of the operator B is a difficult problem to be researched. Nevertheless, for the coarse estimates we can choose $\gamma_1 = 1$, $\gamma_2 = 1 + C_2 + aC_1C_2$, where C_1, C_2 are the constants in (3.26), (3.27). But in their turn, the constants C_1, C_2 are not easy to find, too. Therefore, while there is not any information of γ_1 and γ_2, we propose to use the simple iteration method with $0 < \tau < 2/\|B\|$, for example, $0 < \tau < 2$ because $\|B\| > 1$. Then the iterative process will still converge, but we can not obtain any estimate of the convergence rate of the method (see [13], p. 105-106).

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Tóm tắt. Việc giải các bài toán biên đối với phương trình cấp bốn bằng cách đưa chúng về các bài toán biên đối với các phương trình cấp hai với mục đích sử dụng các thành tựu đã đạt được đối với các phương trình này thu hút sự quan tâm của nhiều nhà nghiên cứu.

Trong bài báo này, nhờ kỹ thuật được phát triển trong các công trình gần đây của mình, chúng tôi đã xây dựng được các phương pháp lặp giải bài toán biên thứ hai đối với phương trình loại song điều hòa. Tốc độ hội tụ của phương pháp được thiết lập.

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